MASTER'S THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Integrable systems on singular symplectic manifolds

Author: Robert Cardona Aguilar

Advisor: Eva Miranda

Department: Departament de Matemàtiques

Academic year: 2017-2018

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

UNIVERSITAT POLITÈCNICA
DE CATALUNYA
BARCELONATECH

Master's thesis

Master's degree in Advanced Mathematics and Mathematical Engineering

Integrable systems on singular symplectic manifolds

Robert Cardona Aguilar

Supervised by Eva Miranda

Barcelona, June 2018

To my beloved family for their support, advice and pride. I am very grateful to my advisor Eva, she gave me a lot of support and has been very attentive since we started working together. Her enthusiasm will always be an inspiration for me. Thanks also to my great laboratory mates Cédric and Roisin, it is very nice and funny to work in their company and share meals with them. At this point they are probably used to those basic mathematics questions I randomly ask them. Talking mathematics with Amadeu was also exciting, and I appreciated a lot his guidance on the last part of the thesis. I would also like to mention Daniel Peralta, whose tutoring in my stay at ICMAT was very important for some ideas developed here. The research in this thesis is supported by the Spanish Ministry of Education under a Beca de Colaboración en Departamentos.

Abstract

Symplectic geometry is a branch of differential geometry that has been developed around the study of Hamilton's equations of motion. The phase space of this dynamical systems has a symplectic structure and studying these manifolds leads to a better comprehension of classical mechanics, in particular of the motion of celestial bodies. A new area that has been developed in the last years is the study of singular symplectic forms. These are interesting, among other things, to model and understand collisions in Hamiltonian systems. Some of these structures can be understood in the context of Poisson geometry, which is a generalization of symplectic geometry, but not all of them. In this thesis we start by reviewing all needed background on symplectic and Poisson geometry. Using a result on differential topology, we provide a new proof of Liouville's theorem for symplectic and Poisson manifolds. Then we focus on folded symplectic manifolds. For these singular symplectic manifolds we define the concept of integrable system, which was not defined before, and prove a singular Arnold-Liouville theorem in this context. As an important intermediate result we prove a Darboux-Carathéodory theorem for folded symplectic manifolds. These manifolds are somehow a dual counterpart to b^m-symplectic manifolds: we relate the existence of action-angle coordinates between this spaces stating a Desingularization theorem using the process of Deblogging. Finally, keeping in mind that main applications are Celestial Mechanics, we analyse the geometric structure of an example of collision in the restricted three body problem. This analysis is done for the first time in this example, and we use the geometric structure obtained to prove a theorem on ejection-collision orbits in a much shorter way than in [17].

Contents

1 Introduction

The topic of integrable systems is at the crossroads of dynamical systems, mathematical physics and Geometry. From the point of Mathematical Physics integrable systems are examples of Hamiltonian Systems which can be integrated by quadratures. From a point of Dynamical Systems they are important as they provide the starting point to the study of perturbation theory of Hamiltonian systems via the so-called action-angle coordinates (KAM theory). From the point of view of Geometry they are important from two perspectives: On the one hand, they provide natural examples of Lagrangian manifolds and are directly connected to mirror symmetry from the other they are connected to group actions and provide examples of toric manifolds (important from Differential Geometry and Algebraic point of views).

The natural framework to consider integrable systems is that of symplectic manifolds as they are naturally modelled on cotangent bundles T^*M naturally endowed with a closed non-degenerate 2-form $-d\lambda$ (where λ is the Liouville 1-form). Integration of Hamiltonian systems is usually considered in this setting where the functions integrating a certain Hamiltonian function H fulfil the integrability condition $\{f_i, f_j\} = 0$ and $f_1 = H$ and $\{\cdot, \cdot\}$ is the standard Poisson bracket.

In general Integrable systems are given by n functionally independent functions on a symplectic manifold such that the associated Poisson bracket can be defined using the Hamiltonian vector fields using the formula $\{f, g\} = \omega(X_f, X_g)$.

More generally we may consider closed 2-forms which are generically symplectic but fail to be symplectic on an hypersurface where they satisfy some additional condition (such as transversality). These structures are called folded-symplectic forms (where the fold corresponds to the hypersurface where the form is no longer non-degenerate) and have been object of study in the recent papers [6] [22].

An important feature of these structures is that they are more common than symplectic structures in the sense there are fewer topological constraints on a manifold to be folded-symplectic than to be symplectic: For instance any 4-dimensional orientable manifold admits a folded symplectic structure (see [6]) but it is not true that any 4-dimensional manifold admits a symplectic structure, for instance a 4-sphere S^4 cannot admit a symplectic structure (since H^2 vanishes) but admits a folded symplectic structure. In higher dimensions any even dimensional manifold admitting a stable almost complex structure admits a folded symplectic structure.

In this thesis we extend the study of integrable systems and a singular Arnold-Liouville theorem to the realm of folded symplectic manifolds. We also connect this result to the theorem of action-angle coordinates on a class of Poisson manifolds called b-Poisson manifolds (see [16]) and present a duality theorem at the end of this thesis that uses deblogging to transform action-angle coordinates in b^{2k+1}-integrable systems to action-angle coordinates in folded integrable systems. This is why in the first chapter of this thesis we present the three worlds, Symplectic, Poisson and folded symplectic manifolds.

The main new results in this Master thesis are the following:

1. A new proof of the Liouville theorem for symplectic and Poisson manifolds. This method is detailed and applied in section 5. This proof uses a result

in differential topology and existence of some 1-forms. It is essentially different from the classical proof, where a torus action is defined using the flow of Hamiltonian vector fields. This result has already been published in the Journal of Geometry and Physics, with reference: R. Cardona, E. Miranda. *Integrable systems and closed one forms.* J. Geom. Phys. 131 (2018), 204-209.

2. A Darboux-Carathéodory theorem for folded symplectic manifolds. This theorem extends a set of commuting and independent functions to a set of coordinates such that the folded form has a very simple expression. This theorem is key to prove the existence of action-angle coordinates.

3. A singular Arnold-Liouville theorem for integrable systems on Folded symplectic manifolds . This theorem is proved in section 7 and describes the semi-local structure around a point on the critical surface of an integrable system in folded symplectic manifolds.

4. A theorem on Desingularization and action-angle coordinates. This theorem relates action-angle coordinates in b^{2k+1}-integrable systems with action-angle coordinates of a related folded symplectic integrable systems. This relation is given by the process of Deblogging, developped in [10].

5. As a final result we study examples in Celestial mechanics where singular symplectic structures appear. We make a geometric analysis of some coordinate changes done in [17] that was never done for this case. We use the geometric structure to prove with different methods a theorem proved in this same paper.

We will send the results of 2,3, 4 and 5 for publication soon.

2 Symplectic geometry and Lie actions review

In this first section we recall some of the main definitions and properties in Symplectic geometry from [5]. With a clear understanding of symplectic manifolds we can later justify how the Poisson geometry theory and folded-symplectic manifolds develop as a generalization of some of the concepts we present here.

2.1 Skew-symmetric bilinear maps

The first step into defining symplectic manifolds is linear algebra. The notion of symplectic bilinear map, which can be represented as matrices, is the first step to the theory of symplectic geometry.

Let V be a vector space over $\mathbb{R}$ of dimension m. A bilinear map $\Omega : V \times V \to \mathbb{R}$ is **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$, for all $u, v \in V$.

Theorem 1. *Let Ω be a skew-symmetric bilinear map on V. Then there is a basis $u_1, ..., u_k, e_1, ..., e_n, v_1, ..., v_n$ of V such that*

$$\begin{aligned} &\Omega(u_i, v) = 0, && \forall i \text{ and } \forall v \in V, \\ &\Omega(e_i, e_j) = \Omega(v_i, v_j) = 0, && \forall i, j \text{ and } , \\ &\Omega(e_i, f_j) = \delta_{ij}, && \forall i, j. \end{aligned}$$

The dimension of V can be written dim $V = 2n+k$. This basis is not unique, and in matrix notation we have

$$\Omega(u, v) = \mathbf{u} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \mathbf{v}.$$

Proof. Choose a basis $u_1, ..., u_k$ of $U := \{u \in V \mid \Omega(u, v) = 0 \ \forall v \in V\}$. Choose a complementary space W to U in V,

$$V = U \oplus W.$$

Take any nonzero $e_1 \in W$. There is $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$. We can assume that $\Omega(e_1, f_1) = 1$. Let

$$\begin{aligned} W_1 &= \text{ span of } e_1, f_1 \\ W_1^{\Omega} &= \{w \in \Omega | \Omega(w, v) = 0 \text{ for all } v \in W_1\}. \end{aligned}$$

Claim. $W_1 \cap W_1^{\Omega} = \{0\}$.

Suppose $v = ae_1 + bf_1 \in W_1 \cap W_1^{\Omega}$.

$$\begin{aligned} 0 &= \Omega(v, e_1) = -b, \\ 0 &= \Omega(v, f_1) = a, \end{aligned}$$

implies $v = 0$.

Claim. $W = W_1 \oplus W_1^{\Omega}$.

Suppose that $v \in W$ has $\Omega(v, e_1) = c$ and $\Omega(v, f_1) = d$. Then

$$v = \underbrace{(-cf_1 + de_1)}_{\in W_1} + \underbrace{(v + cf_1 - de_1)}_{\in W_1^{\Omega}}.$$

Now take e_2. There is $f_2 \in W_1^{\Omega}$ such that $\Omega(e_2, f_2) \neq 0$. Assume that $\Omega(e_2, f_2) = 1$ and let W_2 be the span of e_2, f_2. Etc. This iteration stops because dim $V < \infty$. We obtain then

$$V = U \oplus W_1 \oplus ... \oplus W_n$$

where all the terms are orthogonal with respect to Ω, and W_i has basis e_i, f_i with $\Omega(e_i, f_i) = 1$.

The dimension of the subspace U does not depend on the choice of the basis, so

$$k := \dim U \text{ is an invariant of } (V, \Omega).$$

Since $k + 2n = m = \dim V$, we have that n is an invariant of (V, Ω); $2n$ is called the rank of Ω. $\square$

Definition 1. *The map $\tilde{\Omega} : V \to V^*$ is the linear map defined by $\tilde{\Omega}(v)(u) = \Omega(v, u)$.*

Definition 2. *A skew-symmetric bilienar map Ω is **symplectic** (or nondegenerate) if $\tilde{\Omega}$ is bijective. Then Ω is called a linear symplectic structure on V, and (V, Ω) is called a symplectic vector space.*

As $\tilde{\Omega}$ is bijective, we have by Theorem 1 a basis $e_1, ..., e_n, v_1, ..., v_n$ and we have

$$\Omega(u,v) = [\text{— } u \text{ —}] \begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}.$$

In particular, for a vector space to be symplectic, it has to be of even dimension.

2.2 Symplectic structure on manifolds and Hamiltonian vector fields

With this linear algebra properties in mind, we proceed to define properly the object of study of symplectic geometry: symplectic manifolds.

Definition 3. *Given a manifold M^{2n} of even dimension and a closed non-degenerate 2-form $\omega \in \Omega(M)$, the pair (M^{2n}, w) is called a* ***symplectic manifold****.*

A diffeomorphism f from (M_1, ω_1) to (M_2, ω_2) is a symplectomorphism if $f^*\omega_2 = \omega_1$.

Example. A very important example of symplectic structure can be constructed in the cotangent bundle $M = T^*X$ of any n-dimensional manifold X. Let $(U, x_1, ..., x_n)$ be a coordinate chart at $x \in X$ with $x_i : U \to \mathbb{R}$ the coordinate facts. The differentials $(dx_1)_x, ..., (dx_n)_x$ form a basis of T_x^*X. That means for $\xi \in T_x^*X$, $\xi = \sum_{i=1}^n \xi_i (dx_i)_x$ for some $\xi_i \in \mathbb{R}$. In particular it induces a map

$$\begin{aligned} T^*U &\longrightarrow \mathbb{R}^{2n} \\ (x,\xi) &\longmapsto (x_1, ..., x_n, \xi_1, ..., \xi_n). \end{aligned}$$

This is a coordinate chart for M. The transition functions on intersections are smooth: given two charts $(U, x_1, ..., x_n, \xi_1, ..., \xi_n)$ and $(U', x_1', ..., x_n', \xi_1', ..., \xi_n')$ then

$$\xi = \sum_{i=1}^n \xi_i (dx_i)_x = \sum_{i,j} \xi_i \frac{\partial x_i}{\partial x_j'} (dx_j')_x = \sum_{i=1}^n \xi_i' (dx_i')_x.$$

Hence M is a $2n$-dimensional manifold. We can define on it a 2-form ω by

$$\omega = \sum_{i=1}^n dx_i \wedge d\xi_i.$$

Clearly, defining the 1-form

$$\alpha = \sum_{i=1}^n \xi_i dx_i,$$

we have that $\omega = -d\alpha$. The form ω is independent from coordinates as a consequence of the following claim.

Claim. *The form α is intrinsically defined.*

Proof. Let $(U, x_1, ..., x_n, \xi_1, ..., \xi_n)$ and $(U', x'_1, ..., x'_n, \xi'_1, ..., \xi'_n)$ be two coordinate charts for the cotangent space. On $U \cap U'$, the two sets of coordinates are related by the change of charts $\xi'_j = \sum_i \xi_i(\frac{\partial x_i}{\partial x'_j})$. Since $dx'_j = \sum_i (\frac{\partial x'_j}{\partial x_i})dx_i$, we have

$$\alpha = \sum_i \xi_i dx_i = \sum_j \xi'_j dx'_j = \alpha' \qquad \square$$

It is the tautological form, called the Liouville 1-form, and ω is the canonical symplectic form.

An observation that can be done about symplectic manifolds is that they are necessarily orientable. Consider the $2n$-form ω^n: by definition of ω it never vanishes and so defines a volume form in M (equivalent to M being orientable). In the particular case of dimension 2, a volume form is exactly a symplectic form. This leads to another example of class of symplectic manifolds: **orientable surfaces**. Some other examples are

- $(\mathbb{R}^{2n}, \omega_0 = \sum_{i=1}^n dx_i \wedge dy_i)$,
- $(\mathbb{C}^n, \omega = \frac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i)$,
- $(S^2, \omega = dh \wedge d\theta)$.

The $(\mathbb{R}^{2n}, \omega_0)$ example is very important : any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem (Darboux)**.** *Let (M, ω) be a $2m$-dimensional symplectic manifold and p be any point in M. Then there is a coordinate chart $(U, x_1, ..., x_n, y_1, ..., y_n)$ centered at p such that on U*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

This chart is called a ***Darboux chart.***

Let's recall now a few concepts about vector fields in a manifold M, and introduce new ones for a symplectic manifold (M, ω). Given a complete vector field $X \in \Gamma(M)$, its flow $\varphi_t : M \longrightarrow M$ is the unique solution for any $p \in M$ of the ODE:

$$\begin{cases} \varphi_0(p) = p \\ \frac{\partial \varphi_t}{\partial t}(p) = X(\varphi_t(p)). \end{cases}$$

The uniqueness and existence of the solution is given by Picard theorem. The family $\{\varphi_t | t \in \mathbb{R}\}$ is then called a one-parameter group of diffeomorphisms of M and denoted

$$\varphi_t = \exp tX.$$

Definition 4. *The Lie derivative of a differential form α with respect to the vector field X is:*

$$\mathcal{L}_X \alpha = \frac{d}{dt}({\varphi_t}^* \alpha)|_{t=0}.$$

Definition 5. *A vector field $X \in \Gamma(M)$ is symplectic if $\mathcal{L}_X \omega = 0$.*

Define the interior product or contraction :

$$\begin{aligned} i_X : \Omega^k(M) &\longrightarrow \Omega^{k-1}(M) \\ \alpha &\longmapsto \iota_X \alpha(X_1, ..., X_{k-1}) \\ &\qquad = \alpha(X, X_1, ..., X_{k-1}). \end{aligned}$$

We can state now the **Cartan's formula**, that relates as follows Lie derivative with the interior product and the exterior derivative d,

$$\mathcal{L}_X \omega = (d \circ \iota_X + \iota_X \circ d)\omega. \tag{1}$$

Finally, using Equation (1) and the fact that ω is closed (i.e $d\omega = 0$) we obtain that:

$$\text{X is symplectic} \iff \mathcal{L}_X \omega = 0 \iff \iota_X \omega \text{ is closed .}$$

Note that $i_X \omega$ is a one-form in M. A particular case of closed forms are exact forms, that is $i_X w = d\beta$ for a certain smooth function β in M. Vector fields X such that $i_X \omega$ is exact are called **Hamiltonian** vector fields. In particular, for any smooth function $f : M \longrightarrow \mathbb{R}$ we have by nondegeneracy a unique vector field X_f on M such that $\iota_{X_f} \omega = df$ (depending on the context, it is sometimes defined as satisfying $\iota_{X_f} \omega = -df$). It is called the Hamiltonian vector field with Hamiltonian function f. This also allows to define a bracket of functions in M that we will later analyse, via the formula $\{f, g\} := \omega(X_f, X_g)$.

Example. Consider the symplectic manifold $(\mathbb{R}^4, \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. If we consider for example the function $f = {x_2}^2 + {y_1}^2$ we can compute its corresponding Hamiltonian vector field. The derivative of our function is $df = 2x_2 dx_2 + 2y_1 dy_1$. On the other hand, any vector field is of the form

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + b_1 \frac{\partial}{\partial y_1} + b_2 \frac{\partial}{\partial y_2}, \quad a_1, a_2, b_1, b_2 \in C^\infty(\mathbb{R}^4).$$

Then, the interior product is

$$\begin{aligned} \iota_X \omega &= \omega(X, \cdot) \\ &= dx_1 \wedge dy_1(X, \cdot) + dx_2 \wedge dy_2(X, \cdot) \\ &= a_1 dy_1 + a_2 dy_2 - b_1 dx_1 - b_2 dx_2. \end{aligned}$$

Imposing $\iota_X \omega = df$ we obtain that $a_1 = 2y_1, a_2 = 0, b_1 = 0$ and $b_2 = -2x_2$. The Hamiltonian vector field with Hamiltonian function f is hence

$$X_f = 2y_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial y_2}.$$

Remark. The notion of Hamiltonian vector field is very important and explains the big relation between symplectic geometry with physics. Consider n particles moving in $\mathbb{R}^n$. The phase space is then $\mathbb{R}^{2n}$ with coordinates $(q_1, ..., q_n, p_1, ..., p_n)$, where $\mathbf{q}$ are the position of the particles and $\mathbf{p}$ are the momenta. The manifold is equipped with the canonical symplectic form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$.

Let H be a Hamiltonian in physical terms: it is a function associated to a dynamical system, usually the total energy of the system. Let X be a general vector field in $\mathbb{R}^{2n}$, so of the form

$$X = a_1 \frac{\partial}{\partial q_1} + ... + a_n \frac{\partial}{\partial q_n} + b_1 \frac{\partial}{\partial p_1} + ... + \frac{\partial}{\partial p_n}.$$

Imposing $\iota_X \omega = dH$ we obtain that the hamiltonain vector field X_H is

$$X_H = \frac{\partial H}{\partial p_1}\frac{\partial}{\partial q_1} + ... + \frac{\partial H}{\partial p_n}\frac{\partial}{\partial q_n} - \frac{\partial H}{\partial q_1}\frac{\partial}{\partial p_1} - ... - \frac{\partial H}{\partial q_n}\frac{\partial}{\partial p_n}.$$

The equations of the flow of this vector field are

$$\begin{cases} \dot{q}_i &= \frac{\partial H}{\partial p_i} \;\; i = 1, ..., n \\ \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \;\; i = 1, ..., n. \end{cases} .$$

These are exactly **hamilton's equations of classical mechanics**.

2.3 Lie groups and actions

We present here a very important tool in differential geometry : Lie group theory. This is going to be crucial when understanding integrable systems in a whole different kinds of manifolds.

Definition 6. *G is called a Lie group if G is a smooth manifold and there exist two smooth maps:*

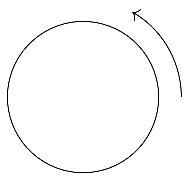
$$\begin{aligned} m : G \times G &\longrightarrow G \\ (x, y) &\longmapsto m(x, y), \end{aligned}$$

and

$$\begin{aligned} i : G &\longrightarrow G \\ x &\longmapsto i(x), \end{aligned}$$

where m is the product and i the inverse giving G a group structure.

Examples. A very simple example of Lie group is $\mathbb{R}$ with addition. Another example of Lie group is the circle S^1 with rotation through θ the standard angle $(mod(2\pi))$. This is equivalent to complex numbers with modulus 1 with multiplication.



Definition 7. *A Lie action φ of G a Lie group on a manifold M is a map :*

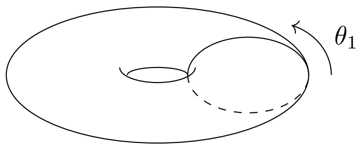
$$\begin{aligned} \varphi : G \times M &\longrightarrow M \\ (g, x) &\longmapsto \varphi(g, x) = g \cdot x, \end{aligned}$$

*such that $e \cdot x = x$ and $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ and φ is smooth. It is **effective** if all element $g \in G \backslash \{e\}$ moves at least one point $p \in M$. It is **free** if e is the only element in G with fixed points.*

Example. We can consider an action of S^1 on $\mathbb{T}^2$ which consist simply on rotating one of the coordinate angles of the torus

$$\begin{aligned} \varphi : S^1 \times \mathbb{T}^2 &\longrightarrow \mathbb{T}^2 \\ (\alpha, (\theta_1, \theta_2)) &\longmapsto (\theta_1 + \alpha, \theta_2). \end{aligned}$$

This action is free and effective.



Note that an action can also be written as $\varphi : G \longrightarrow \text{Diff}(M)$. Now given a Lie group G, we denote T_eG (tangent space at neutral element $e \in G$) as $\mathfrak{g}$. For a given $g \in G$ we can consider the smooth function "left multiplication":

$$\begin{aligned} L_g : G &\longrightarrow G \\ h &\longmapsto g \cdot h, \end{aligned}$$

and consider its differential $dL_g|_e : T_eG \longrightarrow T_gG$.

For a given tangent vector $X \in T_eG$ we define the vector field $\tilde{X} \in \Gamma(G)$ as $\tilde{X}_g = dL_g|_e(X)$ and a bracket in $\mathfrak{g}$:

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto ([\tilde{X}, \tilde{Y}]_G)_e, \end{aligned}$$

where $[\cdot, \cdot]_G$ is the usual Lie bracket for vector fields.

Definition 8. *A **Lie algebra** is a vector space $\mathfrak{g}$ over some field F with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:*

1. *bilinearity,*
2. *skew-symmetry: $[x, y] = -[y, x]$, for all $x, y \in \mathfrak{g}$ and*
3. *Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.*

Definition 9. *With this new defined bracket, $\mathfrak{g}$ has a Lie algebra structure: we call $\mathfrak{g}$ the Lie algebra of the Lie group G.*

Example. A simple example is the case where $G = \mathbb{R}^n$. Then $\mathfrak{g} = T_0\mathbb{R}^n \cong \mathbb{R}^n$. If we take $X \in \mathfrak{g}$, let's check what is $\tilde{X}_g$. By definition $\tilde{X}_g = d(Lg)|_{t=0}(X)$. If we take for example the curve $\gamma(t) = tX$ then

$$\begin{aligned}\tilde{X}_g &= \frac{d}{dt}(Lg \circ \gamma)|_{t=0} \\ &= \frac{d}{dt}(g + tX)|_{t=0} \\ &= X.\end{aligned}$$

As X is fixed, we have $\tilde{X}_g$ constant $\forall g \in G$. Then the bracket of any two tangent vectors X, Y is $[X, Y] = ([\tilde{X}, \tilde{Y}]_G)_0 = (\tilde{X}(\tilde{Y}) - \tilde{Y}(\tilde{X}))_0 = 0$. Last equality stands because constant vector fields commute.

Definition 10. *An action φ is a* ***symplectic action*** *if*

$$\varphi : G \longrightarrow \mathit{Sympl}(M, \omega) \subset \mathit{Diff}(M, \omega),$$

i.e., G 'acts by symplectomorphisms'.

For the special case where the group acting is $\mathbb{R}$, we have a bijection between complete vector fields in M and smooth Lie actions of $\mathbb{R}$ on M given by:

$$\begin{aligned}\{\text{complete vector fields on } M\} &\longleftrightarrow \{\text{Lie actions of } \mathbb{R} \text{ on } M\} \\ X &\longmapsto \exp tX \\ X_p = \frac{d\varphi_t(p)}{dt}|_{t=0} &\longleftarrow\!\shortmid\, \varphi.\end{aligned}$$

Claim. *Symplectic complete vector fields are in a one-to-one correspondence with symplectic actions.*

Proof. If the action is symplectic, we have that $\varphi_t^*\omega = \omega$ $\forall t$. The associated vector field $X_p = \frac{d\varphi_t(p)}{dt}$ is symplectic because

$$\begin{aligned}\mathcal{L}_X\omega &= \frac{d}{dt}(\varphi_t^* w)|_{t=0} \\ &= \frac{d}{dt}(w) \\ &= 0.\end{aligned}$$

In the other way, given a symplectic vector field X, its associated action is $\varphi_t = \exp tX$. As X is symplectic, we have $\mathcal{L}_X\omega = \frac{d}{dt}(\varphi_t^*\omega)|_{t=0} = 0$. In particular $\varphi_t^*\omega$ is constant. For $t = 0$, we have $\varphi_0 = \omega$ so we conclude that $\varphi_t^*\omega = \omega$ i.e. φ_t is a symplectic action. $\square$

In the special case of Hamiltonian vector, we have the following definition:

Definition 11. *A symplectic action φ of $\mathbb{R}$ or S^1 on (M, w) is Hamiltonian if the vector field generated by φ is Hamiltonian i.e. $i_X\omega = dH$ for a certain function H in M.*

For the case where G the group acting on M is an n-torus $\mathbb{T}^n$ then an action of G on M is called Hamiltonian if each restriction

$$\varphi^i := \varphi|_{i\text{th } S^1 \text{ factor}} : S^1 \longrightarrow Sympl(M, \omega)$$

is Hamiltonian in the previous sense.

Example. Let's see a very simple example of Hamiltonian action. Consider the action of translating one coordinate in $\mathbb{R}^{2n}$ with $\omega = \sum dx_i \wedge dy_i$.

$$\varphi(t, (x_1, y_1, ..., x_n, y_n)) = (x_1, y_1 + t, ..., x_n, y_n).$$

The associated vector field is $X = \frac{\partial}{\partial y_1}$. This vector field is Hamiltonian with Hamiltonian function $f = -x_1$. An easy computation checks it:

$$\begin{aligned}\iota_X \omega &= \omega(X, \cdot) \\ &= \sum dx_i \wedge dy_i(X, \cdot) \\ &= dx_1 \wedge dy_1(\frac{\partial}{\partial y_1}) \\ &= -dx_1 \\ &= d(-x_1).\end{aligned}$$

Example. Consider the S^1 action on T^2 seen on a previous example taking the symplectic form $\omega = d\theta_1 \wedge d\theta_2$,

$$\varphi(\psi, (\theta_1, \theta_2)) \longmapsto (\theta_1 + \psi, \theta_2).$$

The vector field generated by this action is $X = \frac{\partial}{\partial \theta_1}$, then we have

$$\begin{aligned}\iota_X \omega &= d\theta_1 \wedge d\theta_2(X, \cdot) \\ &= d\theta_2.\end{aligned}$$

This is obviously a closed form, but it is not exact because θ_2 is not globally defined. It is an example where the action is symplectic but not Hamiltonian.

2.4 Moment map

We denote (M, ω) a compact and connected symplectic manifold, G a Lie group, $\mathfrak{g}$ the associated Lie algebra and $\mathfrak{g}^*$ its dual. We present a pretty technical construction that allows to associate to certain Lie group action a map that describes it in a very simple way: the moment map. This map has some interesting interpretations when analysing an integrable system as we will see later.

Definition 12. *Given a symplectic action (i.e $\varphi^*\omega = \omega$) of G on M , for $X \in \mathfrak{g}$* ***the fundamental vector field*** *$X^{\#}$ associated to X is the vector field in M such that its flow is $exp(uX)$.*

Note now that G acts on itself by conjugation:

$$\begin{aligned}G &\longrightarrow \text{Diff}(G) \\ g &\longmapsto \psi_g(a) = g \cdot a \cdot g^{-1}.\end{aligned}$$

As $\psi_g(e) = e$, its differential at e is an invertible linear map :

$$Ad_g = d\psi_g|_e : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Letting g vary, we obtain the **adjoint representation** of G on $\mathfrak{g}$:

$$\begin{aligned} Ad : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto Ad_g. \end{aligned}$$

From this we define the **coadjoint representation** Ad^* :

$$\begin{aligned} Ad^* : G &\longrightarrow Gl(\mathfrak{g}^*) \\ g &\longmapsto Ad_g^*, \end{aligned}$$

where Ad_g^* is a linear map that goes from $\mathfrak{g}^*$ to $\mathfrak{g}^*$ defined as (note that $Ad_g^*(\xi)$ is a form that acts on tangent vectors in $\mathfrak{g}$):

$$Ad_g^*(\xi)(X) = \langle \xi, Ad_{g^{-1}}(X) \rangle = \xi(Ad_{g^{-1}}(X)).$$

Definition 13. *A moment map associated to the action φ is a map*

$$\mu : M \longrightarrow \mathfrak{g}^* \text{ such that:}$$

1) For all $X \in \mathfrak{g}$ we have $\mu(p)(X) := \langle \mu(p), X \rangle$ and $d\mu(X) = i_{X^\#}\omega$ where $X^\#$ is the fundamental vector field generated $\{\exp(tX) | t \in \mathbb{R}\}$

2) μ is equivariant with respect to the given action φ and the coadjoint action Ad^*, that is:

$$\mu \circ \varphi_g = Ad_g^* \circ \mu, \qquad \forall g \in G.$$

There are particular cases (in which we are mainly interested in this thesis) where these conditions can be rephrased.

Case $G = S^1$ or $G = \mathbb{R}$: we have $\mathfrak{g} \cong \mathbb{R}$, $\mathfrak{g}^* \cong \mathbb{R}$,

1) for the generator $X = 1$ of $\mathfrak{g}$, we have $\mu(p)(X) = \mu(p).1$, i.e. $\mu(X) = \mu$, and $X^\#$ is the standard vector field on M generated by S^1. Then $d\mu = i_{X^\#}\omega$.

2) μ is invariant: $\mathcal{L}_{X^\#}\mu = \iota_{X^\#}d\mu = 0$.

Case $G = \mathbb{T}^n$: we have $\mathfrak{g} \cong \mathbb{R}^n$, $\mathfrak{g}^* \cong \mathbb{R}^n$,

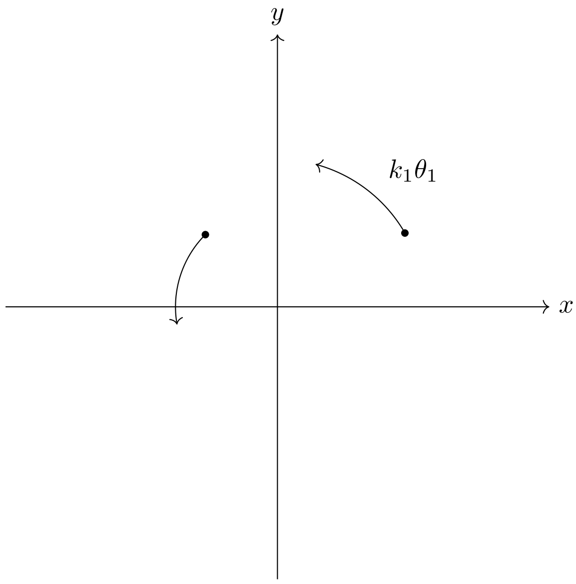
1) For each basis vector X_i of $\mathbb{R}^n$, μ^{X_i} is a Hamiltonian function for $X_i^\#$.

2) μ is invariant i.e. $\iota_{X_j^\#}d\mu_i = 0 \ \forall i, j$.

Example. Let T^n be a n-dimensional torus acting on $\mathbb{C}^n$ by

$$(e^{it_1}, ..., e^{it_n}) \cdot (z_1, ..., z_n) = (e^{it_1k_1}z_1, ..., e^{it_1k_n}z_n),$$

where $k_1, ..., k_n \in \mathbb{Z}$ are fixed. For $n = 1$ this corresponds to a rotation in the complex plane with a speed coefficient k_1.



Given a tangent vector in $\mathfrak{g} \cong \mathbb{R}^n$, $X = A_1 \frac{\partial}{\partial t_1}|_p + ... + A_n \frac{\partial}{\partial t_n}|_p$, the associated fundamental vector field is

$$X^\# = A_1 k_1 \frac{\partial}{\partial \theta_1} + ... + A_n k_n \frac{\partial}{\partial \theta_n}.$$

We took polar coordinates in $\mathbb{C}^n$, $(r_1, \theta_1, ..., r_n, \theta_n)$ and standard symplectic form in polar coordinates $\omega = \sum_{i=1}^n r_i dr_i \wedge d\theta_i$. We can check this formula applying change of coordinates to the differential form ω.

$$\begin{aligned} z_i = r_i e^{i\theta_i} &\implies dz_i = e^{i\theta_i} dr_i + i r_i e^{i\theta_i} d\theta_i, \\ \bar{z}_i = r_i e^{-i\theta_i} &\implies d\bar{z}_i = e^{-i\theta_i} dr_i - i r_i e^{-i\theta_i} d\theta_i. \end{aligned}$$

Then

$$\begin{aligned} \tfrac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i &= \tfrac{i}{2} \sum_{i=1}^n (e^{i\theta_i} dr_i + i r_i e^{i\theta_i} d\theta_i) \wedge (e^{-i\theta_i} dr_i - i r_i e^{-i\theta_i} d\theta_i) \\ &= \tfrac{i}{2} \sum_{i=1}^n -2 i r_i dr_i \wedge d\theta_i \\ &= \sum_{i=1}^n r_i dr_i \wedge d\theta_i. \end{aligned}$$

We can compute now the interior product of the fundamental vector field

$$\begin{aligned} i_{X^\#} \omega &= -\sum_{i=1}^n A_i k_i r_i dr_i \\ &= -\tfrac{1}{2} \sum_{i=1}^n A_i k_i d(r_i^2). \end{aligned}$$

The moment map is

$$\begin{aligned}\mu : \mathbb{C}^n &\longrightarrow \mathbb{R}^n \\ (z_1, ..., z_n) &\longmapsto -\tfrac{1}{2}(k_1|z_1|^2, ..., k_n|z_n|^2) \quad (+\text{constant})\end{aligned}$$

since

$$\begin{aligned}d\mu(X) &= d(\langle \mu(p), X\rangle) \\ &= d(-\tfrac{1}{2}(A_1k_1|z_1|^2, ..., A_nk_n|z_n|^2)) \\ &= -\tfrac{1}{2}\sum_{i=1}^{n} A_ik_id(r_i^2).\end{aligned}$$

Example. Take now the complex projective space $\mathbb{CP}^n$. It is defined as the projectivization of $\mathbb{C}^{n+1}$.

$$\mathbb{CP}^n := (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^* = S^{2n+1}/S^1$$

where $\mathbb{C}^*$ acts of $\mathbb{C}^{n+1}\backslash\{0\}$ by component wise multiplication:

$$\lambda(z_0, ..., z_n) = (\lambda z_0, ..., \lambda z_n).$$

Spheres are seen as unit-norm elements of $\mathbb{C}^{n+1}$ and $\mathbb{C}$. This space has symplectic structure (see [29]) which is in fact a consequence of $\mathbb{CP}^n$ being a Kähler manifold (a certain type of differential manifolds with additional structures). The symplectic form is called **Fubini-Study form** and is given by

$$\omega_{FS} = \frac{i}{2|z|^4}\sum_{j,k=1}^{n} |z_j|^2 dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k.$$

This is in fact a 2-form on $\mathbb{C}^{n+1}\backslash\{0\}$, and ω_{FS} is its pullback to the quotient manifold $\mathbb{CP}^n$.

Take the following $\mathbb{T}^1 = S^1$ action on $\mathbb{CP}^1$:

$$e^{it_1} \cdot [z_0, z_1] = [z_0, e^{it_1}z_1],$$

Lets compute the moment map of this action. The action is in fact induced by the one considered before on C^1, with $k_1 = 1$. Taking the affine chart $U_0 = \{[z_0, z_1] \in \mathbb{CP}^1 \;\; |z_0 \neq 0\}$, the Fubiny-Study form is given by [5]:

$$\omega_{FS} = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2}.$$

Where we the coordinates are $\frac{z_1}{z_0} = z = x + iy$. First, we compute the vector field associated to the action. In polar coordinates it is trivial, since we have the action written like this:

$$\varphi : (t_1, (r, \theta)) \longmapsto (r, \theta + t_1).$$

We have then that $\frac{d(f\circ\varphi_t)}{dt}|_{t=0} = \frac{\partial f}{\partial \theta}$. We deduce that the vector field in polar coordinates is $X_{polar} = \frac{\partial}{\partial \theta}$. In particular, changing coordinates to cartesian, we obtain:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

To find the moment map, we have to impose the condition $i_X \omega_{FS} = d\mu$.

$$
\begin{aligned}
i_X \omega_{FS} &= \frac{dx \wedge dy}{(x^2+y^2+1)^2}(X, \cdot) \\
&= \frac{dx \wedge dy}{(x^2+y^2+1)^2}(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \cdot) \\
&= -\frac{xdx}{(x^2+y^2+1)^2} - \frac{ydy}{(x^2+y^2+1)^2}.
\end{aligned}
$$

Taking into account that $x^2 + y^2 + 1 = |z_1|^2 + |z_0|^2 = |z|^2$ where we took $z_0 = 1$ we deduce that

$$
\mu[z_0, z_1] = -\frac{1}{2}(\frac{|z_1|^2}{|z|^2}).
$$

We can check easily that this map satisfies the condition in our chart U_0:

$$
\begin{aligned}
d\mu &= -\frac{1}{2}d(\frac{x^2+y^2}{x^2+y^2+1}) \\
&= -\frac{1}{2}(\frac{2x(x^2+y^2+1)-(x^2+y^2)2x}{(x^2+y^2+1)^2}dx \\
&+ \frac{2y(x^2+y^2+1)-(x^2+y^2)2y}{(x^2+y^2+1)^2}dy \\
&= -\frac{1}{2}(\frac{2x}{(x^2+y^2+1)^2}dx + \frac{2y}{(x^2+y^2+1)^2}) \\
&= -\frac{xdx}{(x^2+y^2+1)^2} - \frac{ydy}{(x^2+y^2+1)^2}.
\end{aligned}
$$

This example generalizes to $\mathbb{CP}^n$ and the action of $\mathbb{T}^n$:

$$
(e^{it_1}, ..., e^{it_n}) \cdot [z_0 : z_1 : ... : z_n] = [z_0 : e^{it_1}z_1 : ... : e^{it_1}z_n],
$$

and has moment map

$$
\mu[z_0, ..., z_n] = -\frac{1}{2}(\frac{|z_1|^2}{|z|^2}, ..., \frac{|z_n|^2}{|z|^2}).
$$

2.5 A topological constraint to symplectic structures

As always when considering certain type of manifolds, an interesting field of study is to look for their topological invariants. We state here the first topological constraint for the existence of a symplectic structure.

Claim. *A compact symplectic manifold (M, ω) has a non-trivial $H^2(M)$, the second cohomology group.*

Proof. One of the consequences of De Rham theorem, a very important result in smooth manifold theory, is that the dimension of $H^n_{DR}(M^{2n}, \mathbb{R})$ is the same as the dimension of $H^n(M, \mathbb{Z})$. Lets show $H^2_{DR}(M, \mathbb{R})$ is not trivial in a symplectic manifold.

As a first observation, note that since ω is closed, so is ω^k, for all $k \in 2, ..., n$. Suppose now $[\omega] = [0]$, i.e. there exists a one-form such that $\omega = d\alpha$. Then using the properties of the exterior derivative and Stokes theorem we have

$$\int_M \omega^n = \int_M d\alpha \wedge \omega^{n-1} = \int_M d(\alpha \wedge \omega^{n-1}) = \int_{\partial M} \alpha \wedge \omega^{n-1} = 0.$$

Last equality stands because M has no boundary because it is compact. But this is a contradiction since ω^n is a volume form so $\omega^n > 0$ which implies $\int_M \omega^n > 0$. We conclude that $[\omega] \neq [0]$ and hence $H^2(M, \mathbb{Z}) \neq 0$. □

Corollary. *The sphere S^{2n} is not symplectic for $n \geq 2$.*

Proof. We know that S^{2n} is compact and that $H^2(S^{2n}) = 0$ for all $n \geq 2$. Hence, S^{2n} is not symplectic. □

3 Poisson Geometry

We head now understanding Poisson geometry. The idea behind Poisson geometry is generalizing some properties of the symplectic form into a wider set of structures. We use in this section notes from the Master course "Differentiable Manifolds" of Eva Miranda, her course in Paris on "Geometry and Dynamics of Singular Symplectic Manifolds" and the lectures of Fernandes and Marcut [12].

3.1 Poisson structure

As we saw, the symplectic form defines what is called a Poisson bracket. This is a general definition.

Definition 14. *A C^∞ smooth Poisson structure is a $\mathbb{R}$-bilinear operation $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying:*

1. *Skew-symmetry: $\{f, g\} = -\{g, f\}$,*
2. *Leibniz rule: $\{f, g.h\} = \{f, g\}.h + g.\{f, h\}$ and*
3. *Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.*

A pair $(M, \{\cdot, \cdot\})$ is called a Poisson manifold with Poisson bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$.

Examples.

- The canonical Poisson bracket in $\mathbb{R}^{2n}$ with coordinates $(x_1, ..., x_n, y_1, ..., y_n)$ is
$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}.$$
- In $\mathbb{R}^2$ we can define the bracket $\{f, g\} = H(x, y).\{f, g\}_{can}$ which is also a Poisson structure.
- Of course the motivating example works well. If we take any symplectic manifold (M, ω), it's Poisson bracket $\{f, g\} = \omega(X_f, X_g)$ defines a Poisson structure on M.

- Take $M = \mathfrak{g}^*$ the dual of a Lie algebra. Then for $\varphi \in \mathfrak{g}^*$ we define

$$\{f, g\}(\varphi) = \langle \varphi, [df_e, dg_e] \rangle.$$

In general, a Poisson structure can be given by a bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$, this comes by defining $\Pi(df, dg) := \{f, g\}$. Conversely for any of these bivector-fields, the bracket formula $\{f, g\} := \Pi(df, dg)$ is skew-symmetric and satisfies Leibniz rule. But if we want that the bracket satisfies also the Jacobi identity then we need an extra condition for the bivector field. In order to understand this condition we first define the **Schouten bracket**, which is a generalization of Lie bracket. For and extended explanation see [12].

Definition 15. *Let $X \in \mathfrak{X}^k(M)$ and $Y \in \mathfrak{X}^l(M)$ be multivector fields. The Schouten bracket of X and Y is the multivector field $[X, Y] \in \mathfrak{X}^{k+l-1}$ defined by:*

$$[X, Y] = X \circ Y - (-1)^{(k-1)(l-1)} Y \circ X,$$

where we have set :

$$Y \circ X(df_1, ..., df_{k+l-1}) := \sum_{\sigma} (-1)^{\sigma} \bar{Y}(\bar{X}(f_{\sigma(1)}, ..., f_{\sigma(k)}), f_{\sigma(k+1)}, ..., f_{\sigma(k+l-1)}),$$

and the sum is over all $(k + l - 1)$-shuffles.

This bracket satisfies some properties that makes easier to compute it which are the following.

Theorem 2 (Schouten-Nijenhuis)**.** *Let A, B and C be multivector fields of degree a, b and c respectively. The Schouten bracket as defined above satisfies:*

1. *Graded anti-commutativity* $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$.
2. *Graded Leibniz rule*
$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C].$$
3. *Graded Jacobi identity*
$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(b-1)(c-1)}[C, [A, B]] = 0.$$
4. *If X is a vector field then $[X, B] = L_X B$.*

Observe now that for a bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$ with its associated bracket $\{f, g\} = \Pi(df, dg)$ we have by the formula of the Schouten bracket:

$$\frac{1}{2}[\Pi, \Pi](df, dg, dh) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$

We deduce that the Jacobi identity is equivalent to the equation $[\Pi, \Pi] = 0$.

Proposition 3. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. Then the associated bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$ satisfies:*

$$[\Pi, \Pi] = 0.$$

Conversely, every bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$ satisfying this equation defines a Poisson bracket by $\{f, g\} := \Pi(df, dg)$.

Let's apply this proposition to determine when a concrete family of bivector fields define a Poisson structure.

Example. Consider in $\mathbb{R}^4$ with coordinates (x_1, y_1, x_2, y_2) the bivector field

$$\Pi_f = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}.$$

Let's find what are the conditions on f in order to have a Poisson structure defined by Π_f. As seen in the last result, we need to impose that $[\Pi, \Pi] = 0$. This way we can also show some computations using the properties satisfied by the Schouten bracket.

$$\begin{aligned}
[\Pi_f, \Pi_f] &= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}\right] \\
&= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x1}\right] + 2\left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}\right] \\
&+ \left[f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}, f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}\right].
\end{aligned}$$

We will study each of the three terms on its own, call them respectively A_1, A_2 and A_3. The idea when computing Schouten bracket is to apply as much as possible the graded Leibniz rule until we reduce the computations to usual Lie brackets of vector fields. Let's comput the first term with all details:

$$\begin{aligned}
A_1 &= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x1}\right] \\
&= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}\right] \wedge \frac{\partial}{\partial y_1} - \frac{\partial}{\partial x_1} \wedge \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\right] \\
&= -\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}\right] \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_1} \wedge \left[\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}\right] \\
&= -\left(\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right] \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_1} \wedge \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right]\right) \wedge \frac{\partial}{\partial y_1} \\
&+ \frac{\partial}{\partial x_1} \wedge \left(\left[\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}\right] \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_1} \wedge \left[\frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1}\right]\right) \\
&= 0.
\end{aligned}$$

For the last equality we used that the usual Lie bracket of vector fields of two different coordinates is zero. We compute now the other two terms omitting some steps.

$$\begin{aligned}
\frac{A_2}{2} &= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, f \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}\right] \\
&= \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, f \frac{\partial}{\partial x_2}\right] \wedge \frac{\partial}{\partial y_2} - f \frac{\partial}{\partial x_2} \wedge \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right] \\
&= -\left(\left[f \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right] \wedge \frac{\partial}{\partial y_1} + \frac{\partial}{\partial x_1} \wedge \left[f \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}\right]\right) \wedge \frac{\partial}{\partial y_2} \\
&= \left(\frac{\partial f}{\partial x_1}\right) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2} + \left(\frac{\partial f}{\partial y_1}\right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial y_2}{\partial.}
\end{aligned}$$

And the last term

$$\begin{aligned} A_3 &= \left[f\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}, f\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2} \right] \\ &= 0. \end{aligned}$$

Once used to the properties of the Schouten bracket, this last term is trivially zero since the bivectorfields we are computing it's bracket only have terms for two of the variables. Since the result of the Schouten bracket of two bivector fields is a 3-vector field, with only two variables it is always going to be zero.

The only problem that can appear is for the second term. We deduce the following result

$$\Pi_f \text{ defines a Poisson structure } \iff f \text{ depends only on } (x_2, y_2).$$

Hamiltonian dynamics. The definition of Poisson manifold can now by rephrased using the characterization with a bivector field by giving a pair (M, Π) where M is a smooth manifold and $\Pi \in \Gamma(\bigwedge^2 TM)$ is a bivector field satisfying $[\Pi, \Pi] = 0$. The next step is to generalize the notion of Hamiltonian vector field for symplectic manifolds in the Poisson case.

Definition 16. *Let (M, Π) be a Poisson manifold. A vector field $X \in \mathfrak{X}(M)$ is a Poisson vector field if $[\Pi, X] = 0$.*

This last definition is equivalent to saying that the Poisson structure is invariant by the flow of the vector field. This is the analogous concept to symplectic vector field. Generalizing now Hamiltonian vector fields:

Definition 17. *Let (M, Π) be a Poisson manifold. For any smooth function $f \in C^\infty(M)$ we associate a vector field X_f called the **Hamiltonian vector field** of f, by setting*

$$X_f(h) := \{f, h\} = \Pi(df, dh).$$

Any Hamiltonian vector is Poisson, since we have $[\Pi, \Pi] = 0$.

Example. In $SO(3)^*$ let $\Pi = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. Coordinate by coordinate we have

$$\begin{cases} \{x^2 + y^2 + z^2, x\} & = 0 \\ \{x^2 + y^2 + z^2, y\} & = 0 \\ \{x^2 + y^2 + z^2, z\} & = 0 \end{cases}.$$

We can then deduce that $X_{x^2+y^2+z^2} = 0$. This is a function that has a zero Hamiltonian vector field associated. Such functions are called **Casimir functions**.

Example. Restrict now the Poisson structure in $\mathbb{R}^n$, associated with a matrix A which is skew-symmetric to the open set $\mathbb{R}^n_+ = \{(x_1, ..., x_n) : x^i > 0\}$. Consider the Hamiltonian function $H = \sum_{i=1}^n q_i \log x_i - x_i$, where q_i are fixed real numbers. The we obtain the following equations for the orbits of X_H:

$$\dot{x_i} = \{H, x_i\} = B_i x_i + \sum_{i=1}^n a_{ij} x_i x_j,$$

where $B_i = \sum_{i=1}^n a_{ji} q_j$ are constants. These equations are the Lokta-Volterra equations which are a model for dynamics of the population of n biological species interacting in an ecosystem.

Example. Let now $\mathfrak{g} = \mathfrak{so}(3)$ be the Lie algebra of skew-symmetric 3 dimensional square matrices. It can be identified with $\mathbb{R}^3$ and the Lie bracket is identified with the vector product $\times$. On $\mathfrak{so}(3)^*$ the Poisson bracket, given $v = (x, y, z)$, is given by :

$$\{f, g\}(v) = (\nabla f(v) \times \nabla g(v)).v = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ x & y & z \end{vmatrix}.$$

Take now the Hamiltonian function $H = \frac{x^2}{2I_x} + \frac{y^2}{2I_y} + \frac{z^2}{2I_z}$, then we obtain the equations for the flow of X_H:

$$\begin{cases} \dot{x} & = \{H, x\} = \frac{I_y - I_z}{I_y I_z} yz \\ \dot{y} & = \{H, y\} = \frac{I_z - I_x}{I_x I_z} xz \\ \dot{z} & = \{H, z\} = \frac{I_x - I_y}{I_x I_y} xy \end{cases}.$$

These equations are the Euler equations that models the motion of a top in absence of gravity, moving around its center of mass with moments of inertia I_x, I_y and I_z.

In this context, Hamiltonian vector fields satisfy some of the same properties that they had in the symplectic case.

Proposition 4. *Let (M, Π) be a Poisson manifold and take a function $H \in C^\infty(M)$. Then:*

1. *f is a first integral of $X_H \iff \{H, f\} = 0$,*
2. *H is always a first integral of X_H,*
3. *If f_1 and f_2 are first integrals of X_H then $\{f_1, f_2\}$ is also one.*

Proof. Part 1 follows from the definition of X_H. Indeed if f is a first integral of X_H then $X_H(f) = 0$ and $X_H(f) = \Pi/(df, dH) = \{f, H\}$.

Part 2 follows from the fact that $\{\cdot, \cdot\}$ is skew symmetric. Hence $\{H, H\} = 0$ and by 1 we deduce H is a first integral.

Part 3 is proved using 1 and Jacobi identity. Apply Jacobi identity to H, f_1 and f_2:

$$\{H, \{f_1, f_2\}\} + \{f_2, \{H, f_1\}\} + \{f_1, \{f_2, H\}\} = 0$$

□

Since f_1 and f_2 are first integrals, this last equality gives us that $\{f_1, f_2\}$ is also a first integral.

3.2 Poisson and Symplectic structures, local coordinates

As we saw in the first examples any symplectic manifold (M, ω) is a Poisson manifold, the bracket given by the formula $\{f, g\} = \omega(X_f, X_g)$. But Poisson structures are a more general set of structures as we will now see. We can find conditions such that the Poisson structure comes from a symplectic one, which is not true in general. This analysis is detailed in [12].

Let Π be a bivector field. It determines the following map:

$$\begin{aligned} \Pi^\# : \Omega^1 &\longrightarrow \mathfrak{X}(M) \\ \alpha &\longmapsto \iota_\alpha \Pi. \end{aligned}$$

The interior product is a pointwise operation so this map is induced by smooth bundle map that we can denote the same way: $\Pi^\# : T^*M \longrightarrow TM$.

Definition 18. *Let (M, Π) be a Poisson manifold. The **anchor map** is defined as*

$$\begin{aligned} \Pi^\# : T^*M &\longrightarrow TM \\ \alpha &\longmapsto \Pi(\alpha, \cdot). \end{aligned}$$

Pointwise this is written

$$\begin{aligned} \Pi_x^\# : T_x^*M &\longrightarrow T_xM \\ \alpha_x &\longmapsto \iota_\alpha \Pi_x. \end{aligned}$$

Now for a bivector field $\Pi \in \mathfrak{X}^2(M)$ we will say it is non-degenerate at a point $x \in M$ if the anchor map at that point $\Pi_x^\#$ is an isomorphism. If the bivector field is non-degenerate at all points in M then we say that Π is non-degenerate. This is equivalent also to say that Π_x as a skew-symmetric bilinear form is non-degenerate for any point $x \in M$.

With a similar construction if we let ω be a two-form in M, it determines a map $\omega^\flat : TM \to T^*M$ that at any point is

$$\begin{aligned} \omega_x^\flat : T_xM &\longrightarrow T_x^*M \\ v &\longmapsto \iota_v \omega_x. \end{aligned}$$

The following lemma describes the relation between these last two conditions.

Lemma 5. *There is a one-to-one correspondence between non-degenerate bivector fields and non-degenerate 2-forms given by:*

$$\omega^\flat = (\Pi^\#)^{-1} \longleftrightarrow \Pi^\# = (\omega^\flat)^{-1}.$$

Under this correspondence, if Π is associated to ω one has:

$$[\Pi, \Pi](df_1, df_2, df_3) = -2d\omega(\Pi^\#(\alpha), \Pi^\#(\beta), \Pi^\#(\gamma)), \quad \alpha, \beta, \gamma \in T^*M.$$

Proof. The one-to-one correspondence is clear. Let's check the equality, and it is enough to check that it holds for exact one-forms.

$$[\Pi, \Pi](df_1, df_2, df_3) = 2(\{\{f_1, f_2\}, f_3\} + \{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}).$$

Using Cartan's formula for ω:

$$
\begin{aligned}
dw(X,Y,Z) &= X(\omega(Y,Z)) + \text{cycl. perm } X,Y,Z \\
&= -(\omega([X,Y],Z) + \text{cycl. perm } X,Y,Z).
\end{aligned}
$$

Taking now $X = \Pi^{\#}(df_1)$,$Y = \Pi^{\#}(df_2)$ and $Z = \Pi^{\#}(df_3)$ we obtain

$$
d\omega(X,Y,Z) = -(\{\{f_1,f_2\},f_3\} + \{f_2,f_3\},f_1\} + \{\{f_3,f_1\},f_2\}).
$$

□

As a remark, if $dim(M) = 2n$ notice that a bivector field $\Pi \in \mathfrak{X}^2(M)$ is non-degenerate at $x \in M$ if and only if $\wedge^n \Pi_x \neq 0$. The conclusion is

Proposition 6. *There is a one-to-one correspondence between non-degenerate Poisson structures and symplectic structures on a manifold M.*

It becomes now clear that any Poisson structure that is degenerate at some point can not come from any symplectic structure. We can easily find an example of a Poisson bivector field that is not associated to a symplectic form using the example seen in the previous section:

$$
\Pi_f = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + f(x_2,y_2)\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2}.
$$

Taking for instance $f = x_2$, we then have that

$$
\Pi_f \wedge \Pi_f = 2x_2 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y_2},
$$

which is clearly degenerate in $\{x_2 = 0\}$.

Now that we checked that Poisson manifolds is indeed a wider set of structures than symplectic ones, we can study how a Poisson manifold looks locally. Since it is a local structure the support of the bracket of two functions satisfies

Proposition 7. *Let (M,Π) be a Poisson manifold. For any smooth functions $f,g \in C^\infty(M)$ we have*

$$
\operatorname{supp}(\{f,g\}) \subset \operatorname{supp}(f) \cap \operatorname{supp}(g).
$$

Proof. Let $x_0 \notin \operatorname{supp}(f)$. Take the open sets $V := M\backslash\{x_0\}$ and $U := M\backslash\operatorname{supp}(f)$, they cover M. We can then choose a partition of unit $\{p_U,p_V\}$ subordinated to this cover. Computing the bracket at x_0:

$$
\begin{aligned}
\{f,g\}(x_0) &= \{p_U f + p_V f, g\}(x_0) \\
&= \{0 + p_V f, g\}(x_0) \\
&= p_V(x_0)\{f,g\}(x_0) + f(x_0)\{p_V,g\}(x_0) \\
&= 0.
\end{aligned}
$$

This proves $\operatorname{supp}(\{f,g\}) \subset \operatorname{supp}(f)$, and similarly we have $\operatorname{supp}(\{f,g\}) \subset \operatorname{supp}(g)$. □

In particular we can restrict to any open subset U to compute the Poisson bracket, denoting it $\{\cdot,\cdot\}_U$. This leads to a local chart characterization.

Proposition 8. *Let (M,Π) be a Poisson manifold. Let $(U; x_1, ..., x_n)$ be a local chart. Then for any $f, g \in C^\infty(M)$ we have*

$$\{f,g\}_U = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Proof. Observe first that $\{1, f\} = 0$ for any $f \in C^\infty(M)$. This is deduced applying Leibniz rule as following:

$$\{1,f\} = \{1.1, f\} = \{1,f\}1 + 1\{1,f\} = 2\{1,f\}.$$

By linearity we deduce that $\{c, f\} = 0$ for any constant $c \in \mathbb{R}$. Denote $x_{0,i}$ the i-th component of x_0. Consider next the Taylor approximation any function $f \in C^\infty(U)$ up to order 2 around x_0 :

$$f(x) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)(x_i - x_{0,i}) + \sum_{i,j=1}^{n} F_{ij}(x)(x_i - x_{0,i})(x_j - x_{0,j}),$$

for some smooth functions $F_{ij} \in C^\infty(U)$. Hence we find:

$$\begin{aligned}\{f,g\}(x) &= \{f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)(x_i - x_{0,i}) + O(2), g(x_0) + \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x_0)(x_i - x_{0,i}) + O(2)\} \\ &= \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}(x_0) \frac{\partial g}{\partial x_j}(x_0) \{x_i, x_j\}(x) + \sum_{i=1}^{n} H_i(x)(x_i - x_{0,i}).\end{aligned}$$

Evaluating at $x = x_0$ we obtain

$$\{f,g\} = \sum_{i,j=1}^{n} \{x_i, x_j\}(x_0) \frac{\partial f}{\partial x_i}(x_0) \frac{\partial g}{\partial x_j}(x_0).$$

The point x_0 was arbitrary so this completes the proof. □

3.3 Symplectic foliation and splitting theorem

Our goal in a first part of this section is to study the distribution defined by the Hamiltonian vector fields of a given Poisson manifold (M,Π). This distribution is

$$\mathcal{D} = \{X_f | \ f \in C^\infty(M)\}.$$

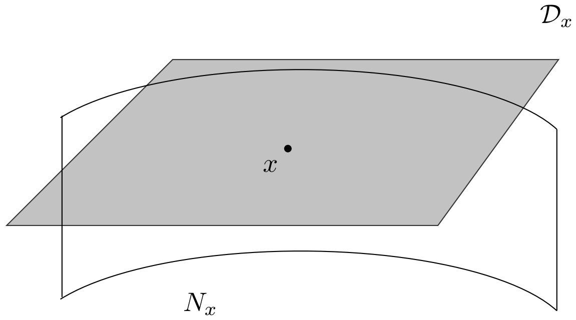
Indeed this distribution can be singular, for example we observed in the previous section that there are functions with zero Hamiltonian vector field. Recall that for a regular distribution we have the notion of integrable distribution and the socalled Frobenius theorem.

Definition 19. *Let M a smooth manifold. A distribution $\mathcal{D} \subset TM$ of rank k is integrable if for any point $p \in M$ there is a submanifold N of dimension k such that $T_pN = \mathcal{D}_p$.*

Then, Frobenius theorem relates this property with involutivity.

Theorem 9. *A regular smooth distribution $\mathcal{D} \in TM$ is integrable if and only if $\mathcal{D}$ is involutive. This means*

$$\text{If } X, Y \in \mathcal{D} \implies [X, Y] \in \mathcal{D}.$$



Let us now take a distribution that may not have constant rank. This is known as a singular distribution and we need to add an extra condition if we want it to be integrable. This condition is called the Stefan's condition, a theorem proved independently by Stefan [25] and by Sussmann [26].

Theorem 10 (Stefan-Sussmann)**.** *A singular,smooth and involutive distribution that satisfies Stefan's condition: $\exists C$ a set of generators of $\mathcal{D}$ such that for all $X \in C$*

$$(\varphi_X^t)_*(\mathcal{D}_p) = \mathcal{D}_{\varphi_X^t(p)},$$

is integrable.

Observe that since the rank of the distribution is not constant, the integral submanifolds will also have different dimensions. We can now apply this last result to prove that the distribution we wanted to study $\mathcal{D} = \{X_f | \ f \in C^\infty(M)\}$ is in fact integrable.

Theorem 11. *In a given Poisson manifold (M, Π), the distribution $\mathcal{D} = \{X_f | \ f \in C^\infty(M)\}$ is integrable.*

Proof. We first show that the distribution is involutive i.e. that for all $X, Y \in \mathcal{D}$ we have that $[X, Y] \in \mathcal{D}$. Observe that

$$\begin{aligned}
[X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) = X_f(\Pi(dg, dh)) - X_g(\Pi(df, dh)) \\
&= \Pi(f, \Pi(dg, dh)) - \Pi(g, \Pi(df, dh)) = \{f, \{g, h\}\} - \{g, \{f, h\}\} \\
&= -\{h, \{f, h\}\} = X_{\{f,g\}}(h).
\end{aligned}$$

In the last line we used Jacobi identity. We note then that the Lie bracket of two Hamiltonian vector fields is also a Hamiltonian vector field which then lies in $\mathcal{D}$. So the distribution is involutive.

It is left now to show that it satisfies the Stefan's condition. But this is directly obtained by the fact that that a Hamiltonian vector field is a Poisson vector field and hence preserves the Poisson structure. In particular it preserves the distribution of Hamiltonian vector fields and we conclude that $\mathcal{D}$ is integrable. $\square$

From now on we will denote this foliation $\mathcal{F}$. We will prove now that it is in fact a symplectic foliation. Recall the anchor map defined before and denoted $\Pi^{\#}$.

$$\begin{aligned} \Pi^* : T^*M &\longrightarrow TM \\ \alpha &\longmapsto \Pi(\alpha, \cdot). \end{aligned}$$

An interesting property of this map is that it satisfies that $\mathcal{D}_x = \text{Im } \Pi^*$. First inclusion is direct since Hamiltonian vectorfields are defined as $X_f = \Pi(df, \cdot) = \Pi^*(df)$. The other one is obtained applying Poincaré's lemma in a neighborhood of any point.

We can now proof that our foliation is indeed symplectic.

Claim. *Let (M, Π) be a Poisson manifold. Then the foliation $\mathcal{F}$ given by the distribution $\mathcal{D} = \{X_f |\ f \in C^\infty(M)\}$ is symplectic.*

Proof. Consider the anchor map Π^*. Since we know that $\mathcal{D}_x = \text{Im } \Pi^*$, for any vector field $X \in \mathcal{D}$ there exists $\alpha \in \Omega^1(M)$ such that $X = \Pi^*(\alpha)$. Consider now the 2-form defined as

$$\Pi^{-1}(X, Y) := \Pi(\alpha, \beta), \quad \text{with } X = \Pi^*(\alpha) \text{ and } Y = \Pi^*(\beta).$$

This form will be the symplectic structure we are looking for.

1. The form is of course skew-symmetric by its definition.

2. Nondegeneracy: suppose there is X such that $\Pi^{-1}(X, Y) = 0$ for all $Y \in \mathcal{D}$. Then

$$\begin{aligned} \Pi^{-1}(X, Y) = 0 &\implies \Pi(\alpha, \beta) = 0 \\ &\implies -\Pi(\alpha, \beta) = -\langle \Pi^*(\beta), \alpha \rangle = 0 \\ &\implies -\alpha(Y) = 0 \implies \alpha = 0 \implies X = 0. \end{aligned}$$

3. Closed form: take three Hamiltonian vector fields $X = X_f = \Pi^*(\alpha), Y = X_g = \Pi^*(\beta)$ and $Z = X_h \Pi^*(\gamma)$. Then

$$\begin{aligned} d\Pi^{-1}(X, Y, Z) &= X(\Pi^{-1}(Y, Z)) - Y(\Pi^{-1}(X, Z)) + Z(\Pi^{-1}(X, Y)) \\ &= X(\Pi(\beta, \gamma)) - Y(\Pi(\alpha, \gamma)) + Z(\Pi(\alpha, \beta)) \\ &= X_f(\{g, h\}) + X_g(\{h, f\}) + X_h(\{f, g\} \\ &= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \\ &= 0. \end{aligned}$$

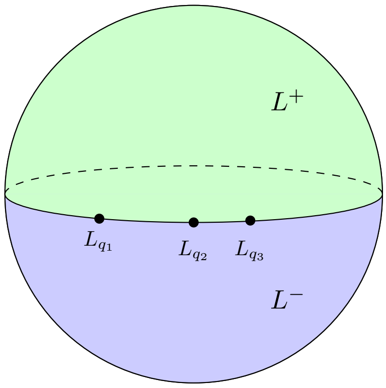
We deduce that $\mathcal{F}$ is a symplectic foliation. □

Examples.

- Take $M = S^2$ with the associated bivector field $\Pi = 0$. Of course we have $\text{Im } \Pi^* = 0$ and all leaves are zero-dimensional i.e. the points of M.

- With the same $M = S^2$, consider now the simplectic form $\omega = dh \wedge d\theta$. The associated bivector field is $\Pi = \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$. In this case we have $\text{Im } \Pi^* = 2$ at all points: there is only one leaf that is the manifold itself.

- Still in $M = S^2$ take now as Poisson bivector field $\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$. The bivector field vanishes at $h = 0$ where it has rank 0. In the sets $\{p \in S^2 | \ h > 0\}$ and $\{p \in S^2 | \ h > 0\}$ the rank is 2. We obtain the following leaves $L^+ = \{p \in M | \ h > 0\}$, $\quad L^- = \{p \in M | \ h < 0\}$ and $L_q = \{q\}$ for any q of the form $q = (0, \theta)$.



We can now understand Poisson manifolds as a combination of symplectic manifolds with different dimensions: as if we had "singularities".

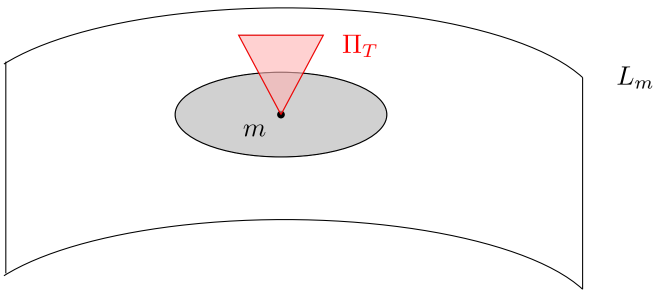
3.4 Weinstein Splitting theorem

The existence of this symplectic foliation can give us an intuition: it may seem that the Poisson bivector field could somehow split into a symplectic part and a trasversal one. This is what the following theorem proves.

Theorem 12 (Weinstein, 1983 Splitting theorem)**.** *Let (M^{2k+s}, Π) be a Poisson manifold and $m \in M$ a point with rank $\Pi(m) = 2k$. Then there exist local coordinates $(x_1, ..., x_k, y_1, ..., y_k, z_1, ..., z_s)$ centered at m with*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{s} \varphi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \quad \textit{with } \varphi_{ij}(0) = 0.$$

The second term is called transversal and usually denoted Π_T.



Proof. We will prove it by induction. The case $k = 0$ is trivial. Suppose now it is true for $k - 1$. Let $m \in (M, \Pi)$ and rank $\Pi_m = 2k > 0$. Since $\Pi_m \neq 0$ there is a function p such that $X_p(m) \neq 0$. Locally we can assume $X_p = \frac{\partial}{\partial q}$ for some coordinates $(q, g_2, ..., g_n)$.

Consider the distribution $\mathcal{D} = \langle X_p, X_q \rangle$. We have that $\{p, q\} = X_p(q) = \frac{\partial}{\partial q}(q) = 1$. In particular $[X_p, X_q] = X_{\{p,q\}} = X_1 = 0$. We deduce that $\mathcal{D}$ is involutive, and of course it is regular. Applying Frobenius theorem there are coordinates $(x_1, y_1, z_1, ..., z_{n-2})$ such that $X_p = \frac{\partial}{\partial x_1}$, $X_q = \frac{\partial}{\partial y_1}$ and $\{p, z_i\} = \{q, z_i\} = 0$. Applying Jacobi identity with p, z_i and z_j

$$\{p, \{z_i, z_j\}\} + \{z_i, \{z_j, p\}\} + \{z_j, \{p, z_i\}\} = 0,$$

we deduce that $\{p, \{z_i, z_j\}\} = 0$. The same way we can obtain that $\{q, \{z_i, z_j\}\} = 0$. Also

$$\begin{aligned} X_p = \frac{\partial}{\partial x_1} &\implies \frac{\partial q}{\partial x_1} = X_p(q) = 1, \; \frac{\partial p}{\partial x_1} = X_p(p) = 0 \\ X_q = \frac{\partial}{\partial y_1} &\implies \frac{\partial q}{\partial y_1} = X_q(q) = 0, \; \frac{\partial p}{\partial y_1} = X_q(p) = -1. \end{aligned}$$

From all these relations it is clear that the change of coordinates $(x_1, y_1, z_1, ..., z_{n-2}) \to (p, q, z_1, ..., z_{n-2})$ has as Jacobian

$$\begin{bmatrix} 0 & 1 & \vec{0} \\ -1 & 0 & \vec{0} \\ \vec{0} & \vec{0} & \mathrm{id} \end{bmatrix}.$$

In this coordinates we have $\Pi = \{p, q\} \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \sum_{i,j} \{z_i, z_j\} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. But since $\{q, \{z_i, z_j\}\} = \{p, \{z_i, z_j\}\} = 0$ we deduce that this is written

$$\Pi = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \sum_{i,j} \varphi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}.$$

The summation is now a Poisson structure of rank $\leq 2k - 2$ and we can apply the induction hypothesis. This proves the theorem. $\square$

3.5 Poisson cohomology

Let (M, Π) be a Poisson manifold. One can construct a cohomology associated to the Poisson bracket as described below. We first define a differential d_Π as following:

$$\begin{aligned} d_\Pi : \Gamma(\bigwedge^k TM) &\longrightarrow \Gamma(\bigwedge^{k+1} TM) \\ X &\longmapsto [\Pi, X]. \end{aligned}$$

Here $[.,.]$ is the Schouten bracket.

Lemma 13. *The differential d_Π satisfies $d_\Pi^2 = 0$.*

Proof. Proving that $d_\Pi^2 = 0$ consists in checking that $\forall A$ multivector field we have that $[\Pi, [\Pi, A]] = 0$. Let k be the degree of A. Using the graded Jacobi identity and graded anti-commutativity we have that:

$$\begin{aligned} &(-1)^{k-1}[\Pi, [\Pi, A]] - [\Pi, [A, \Pi]] + (-1)^{k-1}[A, [\Pi, \Pi]] = 0 \\ \Longrightarrow\ & 2[\Pi, [\Pi, A]] = -[A, [\Pi, \Pi]]. \end{aligned}$$

Since Π is a Poisson strucutre, it satisfies that $[\Pi, \Pi] = 0$ and we deduce that $d_\Pi^2 = 0$ □

We can now define the Poisson cohomology.

Definition 20. *The Poisson cohomology is defined as*

$$H_\Pi^k(M) = \frac{\ker d_\Pi}{\mathrm{Im}\ d_\Pi}.$$

An observation that can be done is that if (M, Π) is a symplectic manifold, then $H_\Pi^k(M) \cong H_{DR}^k(M)$.

4 Folded symplectic manifolds

It is clear that Poisson geometry is a wide generalization of symplectic geometry. However, a simpler way to generalize symplectic forms would be relaxing some of the conditions that we impose to the form. A first approach, as we are going to detail, is to allow the form to go to infinity at a hypersurface of the manifold. This leads to the concept of b-manifold, which happens to be a Poisson manifold. With this same idea, we could now allow the form to vanish at a hypersurface of the manifold: this are folded-symplectic manifolds. However in this case the structure obtained is not a Poisson structure.

4.1 Previous case: b-symplectic

The category of b-manifolds was developped by Melrose [21], in order to study manifolds with boundary. Most of the definitions can be used replacing the boundary by any given hypersurface of the manifold. The definition of these manifolds is the following.

Definition 21. *A b-manifold (M, Z) is an oriented manifold M with an oriented hypersurface Z.*

With it, there is also the notion of b-map.

Definition 22. *A b-map is a map*

$$f : (M_1, Z_1) \longrightarrow (M_2, Z_2)$$

so that f is transverse to Z_2 and $f^1(Z_2) = Z_1$.

Not only maps have to be redefined in the b-category, but also vector fields and differential forms have to be redefined.

Definition 23. *A b-vector field on a b-manifold (M, Z) is a vector field which is tangent to Z at every point $p \in Z$.*

These vector fields form a Lie subalgebra of vector fields on M. Let t be a defining function of Z in a neighborhood U and $(t, x_2, ..., x_n)$ be a chart on it. Then the set of b-vector fields on U is a free $C^\infty(U)$-module with basis

$$\left(t\frac{\partial}{\partial t}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right).$$

We deduce that the sheaf of b-vector fields on M is a locally free C^∞-module and therefore it is given by the sections of a vector bundle on M. We call this vector bundle the b-tangent bundle and denote it ${}^bTM$. It's dual bundle is called the b-cotangent bundle and is denoted ${}^bT^*M$.

In the Poisson we have the notion of b-Poisson manifold which will be in fact the same as b-symplectic manifold.

Definition 24. *Let (M^{2n}, Π) be an oriented Poisson manifold. Let the map*

$$p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$$

be transverse to the zero section. Then Π is called a b-Poisson structure on M. The hypersurface Z where the multivectorfield Π^n vanishes,

$$Z = \{p \in M | (\Pi(p))^n = 0\}$$

is called the critical hypersurface of Π. The pair (M, Π) is called a b-Poisson manifold.

Asking the transversality condition is equivalent to saying that 0 is a regular value of the map $p \longrightarrow (\Pi(p))^n$. The hypersurface Z has a defining function obtained by dividing this map by a non-vanishing section of $\bigwedge^{2n}(TM)$.

By duality with forms and using the b-tangent bundle, one can define b-symplectic manifolds which are in one-to-one correspondence with b-Poisson manifolds.

Definition 25. *Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is b-symplectic if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$.*

This correspondance is proved in [13] and can be formulated as

Proposition 14. *A two-form ω on a b-manifold (M, Z) is b-symplectic if and only if its dual bivector field Π is a b-Poisson structure.*

Let us present some particular examples of this structures.

Example. *Let S be any orientable surface and Z a smooth curve on it. Denote f a defining function of Z and take Ω a volume form on M, with associated Poisson bivector field $\Pi_\Omega = \Omega^*$. Then the Poisson bivector field*

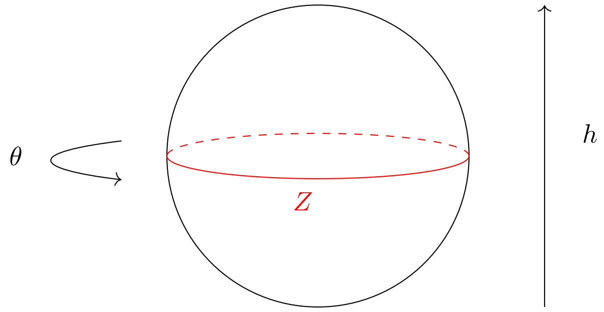
$$\Pi = f\Pi_\Omega$$

is a b-Poisson structure.

Example. *Consider the sphere S^2 with coordinates (h, θ) the height and the standard angle, with the Poisson structure*

$$\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}.$$

In this case, the equator is the critical hypersurface.



Example. Let (N^{2n-1}, Π) a Poisson manifold with rank $2n-2$ rank. Consider then $H = N^{2n-1} \times S^1$ and let X be a Poisson vector field of N. We can find the conditions on f such that

$$\bar{\Pi} = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \Pi$$

is a b-Poisson structure. We first have to check if this defines indeed a Poisson structure:

$$\begin{aligned}
[\bar{\Pi}, \bar{\Pi}] &= \left[f(\theta)\frac{\partial}{\partial\theta} \wedge X + \Pi, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \Pi\right] \\
&= \left[f(\theta)\frac{\partial}{\partial\theta} \wedge X, f(\theta)\frac{\partial}{\partial\theta} \wedge X\right] + 2\left[f(\theta)\frac{\partial}{\partial\theta} \wedge X, \Pi\right] + \left[\Pi, \Pi\right] \\
&= 0,
\end{aligned}$$

where we used that Π is Poisson and that X is Poisson with respect to Π. Now, if we want it to be b-Poisson, we have to impose that $\bar{\Pi}^n$ is transversal to the zero section. If we want this to be true we need to impose that X is transversal to the symplectic foliation and that f vanishes linearly. Then the critical hypersurface consists of the union of as many copies of N as zeros has f.

In this context we have a normal form theorem analogous to Darboux theorem for symplectic manifolds. This results is also proved in [13].

Theorem 15 (**b-Darboux theorem**)**.** *Let (M, Z, ω) be a b-symplectic manifold. Then, on a neighborhood of a point $p \in Z$, there exist coordinates $(x_1, y_1, ..., x_n, y_n)$ centered at p such that*

$$\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Note that with this chart, the symplectic foliation of (M, Π) has an specific form. It has two open subsets where the Poisson structure has maximal rank given by $\{x_1 > 0\}$ and $\{x_1 < 0\}$. The hyperplane $\{x_1 = 0\}$ contains leaves of dimension $2n-2$ given by the level sets of y_1.

4.2 Folded Symplectic: Definition and examples

As we explained in the introduction, another natural generalization of a symplectic form will be when the form vanishes transversally. This leads to the definition of a folded-symplectic form.

Definition 26. *Let M be a $2n$-dimensional manifold. We say that $\omega \in \Omega^2(M)$ is folded-symplectic if*

1. *$d\omega = 0$,*
2. *$\omega^n \pitchfork \mathcal{O}$, where $\mathcal{O} \in \bigwedge^{2n}(T^*M)$ is the zero section, hence $Z = (\omega^n)^{-1}(\mathcal{O})$ is a codimension 1 submanifold,*
3. *$i_Z : Z \to M$ is the inclusion map, ${i_Z}^*\omega$ has maximal rank $2n-2$.*

*We say that (M, ω) is a **folded-symplectic manifold** and we call $Z \subset M$ the folding hypersurface.*

As a comment, observe that the property of being folded is an open property. If ω_0 is folded, a closed 2-form ω that is C^1-close to it is also folded. If M is oriented, then Z acquires a canonical orientation the following way: consider the sets

$$M^+ = \{p \in M | \ \omega_p^n > 0\},$$
$$M^- = \{p \in M | \ \omega_p^n < 0\}.$$

We can write $M \setminus Z = M^+ \cup M^-$. Apply now the following theorem, called tubular neighborhood theorem (see [5]).

Let X be a k-dimensional submanifold of M, with $k < n$ and n the dimension of M. Let $i : X \hookrightarrow M$ be the inclusion map. For any $x \in X$, let $N_x = T_{i(x)}/T_xX$ the normal fiber and denote $NX = \{(x, v)| \ x \in X, v \in N_x\}$ the normal bundle with respect to the embedding i. We let $i_0 : X \to NX$ be the zero section.

Theorem. *Then there exist a convex neighborhood U_0 of X in NX, a neighborhood U of X in M and a diffeomorphism $\varphi : U_0 \to U$ such that the following diagram commutes.*

$$\begin{tikzcd} X \arrow[r, "i_0"] \arrow[d, "i"'] & V \\ U \arrow[ur, "\varphi"'] & \end{tikzcd}$$

One gets this way the normal bundle of Z and hence an orientation of Z itself.

Just as symplectomorphisms in the symplectic case, there is a notion of isomorphism of folded-symplectic manifolds.

Definition 27. *Let (M, ω_1) and (N, ω_2) be two folded-symplectic manifolds. A smooth map $\phi : M \to N$ is folded-symplectic if $\phi^*\omega_2 = \omega_1$. If it is also a diffeomorphism, we say it is a folded-symplectomorphism.*

Remark. Recall that in the b-symplectic case the form dual to the bivector was in fact not defined: it had a term that went to infinity at the hypersurface. Analogously, if we try to obtain the dual bivectorfield of a folded-symplectic form it happens to be non-defined near the critical hypersurface. This is the idea behind the fact that a folded-symplectic structure is not a Poisson structure.

When we are in a neighborhood of M far from Z, locally the manifold is just as a symplectic manifold. Interesting things happen indeed when we are in a neighborhood of a point in Z. Two important vector subbundles can be defined the following way via the inclusion of ω in Z:

Definition 28. *Let (M, ω) be a folded-symplectic manifold and Z the folding hypersurface with inclusion $i_Z : Z \to M$. Assume Z is nonempty.*

1. *$ker(\omega) \to Z$ is a 2-plane bundle over Z whose fiber at a point $z \in Z$ is $ker(\omega_z) = \{X \in T_z Z | i_X \omega_z = 0\}$.*
2. *$ker(i_Z{}^*\omega) \to Z$ the rank 1 vector bundle over Z, that can be defined also as the intersection $ker(\omega) \cap TZ$.*

Observe that $ker(i_Z{}^*\omega)$ is a rank 1 vector subbundle of TZ so it is obviously involutive. Applying Frobenius theorem we obtain a foliation of Z by 1-dimensional leaves.

Definition 29. *Let (M, ω) be a folded-symplectic manifold with nonempty folding hypersurface Z. The foliation $\mathcal{F}$ defined by the rank 1 vector subbundle $ker(i_Z{}^*\omega)$ is called the* ***null-foliation****.*

Let us analyse some examples of folded-symplectic manifolds.

Example. Consider $\omega \in \Omega^2(R^{2n})$ defined by

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

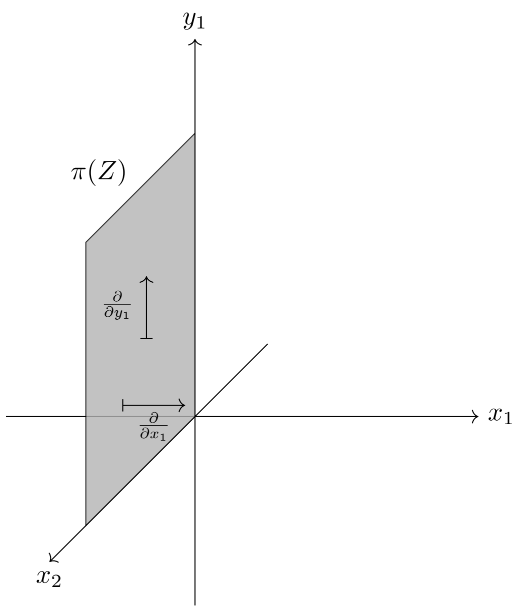
This is a folded symplectic form with folding hypersurface $Z = \{x| \ x_1 = 0\}$. The bundles defined in Definition 3 are, in this case:

1. $ker(\omega) \to Z$ is framed by the vector fields $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\}$ along Z.
2. $ker(i_Z{}^*\omega)$ is framed by the vector field $\frac{\partial}{\partial y_1}$.

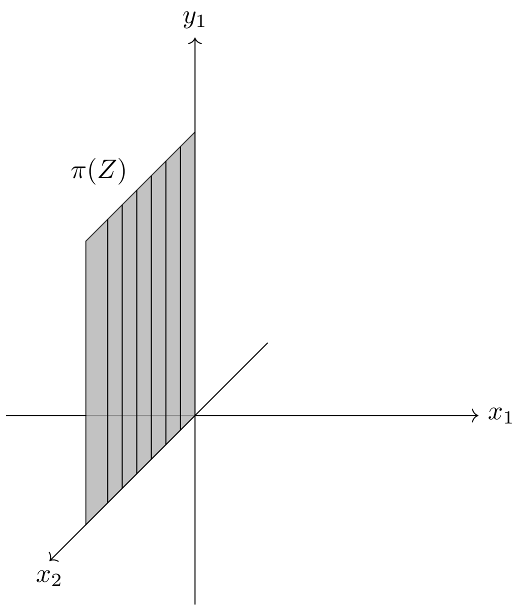
The leaves of the null-foliation are given by fixing coordinates x_i, y_i for $i = 2, ..., n$ and letting vary y_1. The sets that give the canonical orientation to Z are

$$M^+ = \{x \in \mathbb{R}^4 | \ x_1 > 0\},$$
$$M^- = \{x \in \mathbb{R}^4 | \ x_1 < 0\}.$$

When $n = 2$, we can draw the situation of the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ to the coordinates $\{x_1, y_1, x_2\}$.



We can draw a few leaves of the resulting nullfoliation too.



Example. [6] Regard the even-dimensional sphere S^{2n} as the set of unit vectors in $\mathbb{R}^{2n+1}$. Restricting the usual folded symplectic form in $\mathbb{R}^{2n+1}$

$$dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n,$$

to S^{2n} we obtain a folded symplectic form. The folding hypersurface is the equator given by $S^{2n} \cap \{x_{n+1} = 0\}$.

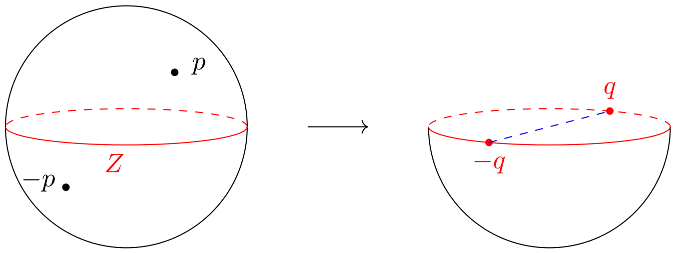
Another way this can be obtained by taking two $2n$-dimensional disks with standard symplectic forms at their boundaries. Reversing the orientation of one of them you obtain the same folded symplectic manifold. Finally, there is another way to obtain it that justifies the name "folded". For this, consider the folding map from the sphere to the disk $\pi : S^{2n} \longrightarrow D^{2n}$, folding along the equator. If we let $\gamma = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$ be the standard symplectic form on the disk, then $\omega = \pi^*\gamma$ is a folded symplectic form on S^{2n}. This is already interesting since in the symplectic case the only sphere that admitted a symplectic structure was S^2.

The manifold can also be non-orientable such as the following example. This is also different than the symplectic case, where the manifold had to be orientable.

Example. Consider S^2 with the folded symplectic form $\omega = hdh \wedge d\theta$. The antipodal map

$$\varphi : (h, \theta) \longmapsto (-h, \theta + \frac{\pi}{2})$$

leaves invariant ω. Hence the form descends by the quotient and we obtain a folded symplectic form in $\mathbb{RP}^2$.



A neighborhood of Z is diffeomorphic to a Moebius band.

Example. In the general case S^{2n} we the folded symplectic structure as detailed two examples above. This form is invariant under the antipodal map $x \mapsto -x$. Hence, it induces a folded symplectic form on $\mathbb{RP}^{2n}$ which is non orientable.

The connected sum also gives a source of examples of folded-symplectic manifolds.

Example. [6] Take now two compact symplectic manifolds (M_1, ω_1) and (M_2, ω_2) of dimension $2n$ and with induced orientations. Consider

$$M = M_1 \# \bar{M}_2$$

the connected sum, where $\bar{M}_2$ has the opposite of the symplectic orientation. Then M has a folded symplectic form that coincides with ω_1 and ω_2 outside of a tubular neighborhood of the surgery. Consider $A_i \cong S^{2n-1} \times I$ two small annuli where the surgery occurs. The symplectic form ω_i restricted to A_i is diffeomorphic to $d(r_i \wedge \pi^*\alpha)$, where r_i is a coordinate on I, π is the projection $S^{2n-1} \times I \to S^{2n-1}$ and α is the standard contact one-form in S^{2n-1}.

Choose coordinates t_1, t_2 such that $r_i = 1 + t_i^2$ for $t_i > \epsilon$. Finally, extend ω across the connected sum by defining it to be

$$\omega = d[(1 + t^2) \wedge \pi^*\alpha],$$

where $t = -t_1$ on the interval $t < -\epsilon$ and $t = t_2$ on the interval $t > \epsilon$. The folding hypersurface of ω is given by $t = 0$.

4.3 Normal forms

For the normal form of a folded symplectic manifold we will follow the proof in [22] that uses results in [6]. Let (M, ω) be an oriented folded-symplectic manifold with folding hypersurface Z. Call V the line field defining the nullfoliation on Z. If we denote E the rank 2 bundle over Z whose fiber at each point is the kernel of ω (see Definition 3) then $V = E \cap TZ$. The form ω^{n-1} gives an orientation of $(i^*TM)/E$ which induces an orientation on E. Finally, orientations of E and TZ induce an orientation on the nullfoliation.

Let v be an oriented non-vanishing section of V and $\alpha \in \Omega^1(M)$ a one-form such that $\alpha(v) = 1$.

Proposition 16. *Assume Z is compact. Then there is a tubular neighborhood U of Z in M and an orientation preserving diffeomorphism $\varphi : Z \times (-\epsilon, \epsilon) \to U$ mapping $Z \times \{0\}$ onto Z such that*

$$\varphi^*\omega = p^*i^*\omega + d(t^2 p^*\alpha),$$

where $p : Z \times (-\epsilon, \epsilon) \to Z$ is the projection onto the first factor and t is the real coordinate in $(-\epsilon, \epsilon)$.

Proof. Let w be a vector field on M such that for all $z \in Z$ we have that (w_z, v_z) is an oriented basis of E_z. Let U be a tubular neighborhood of Z in M and $\rho : Z \times (-\epsilon, \epsilon) \to U$ the map that takes $Z \times \{0\}$ onto Z and the lines $\{z\} \times (-\epsilon, \epsilon)$ onto the integral curves of w. Using ρ we can identify U with $Z \times (-\epsilon, \epsilon)$ and w with $\frac{\partial}{\partial t}$. Furthermore, ρ allows us to extend the vector field v to all U by the inclusion $T_z Z$ into $T_{(z,t)}U$.

The idea is to apply the "Moser trick" to the forms $\omega_0 := p^*i^*\omega + d(t^2 p^*\alpha)$ and $\omega_1 := \omega$ by setting $\omega_s := (1-s)\omega_0 + s\omega_1$ and finding a vector field v_s on M such that

$$\mathcal{L}_{v_s}\omega_s + \frac{d\omega_s}{ds} = 0.$$

We need this lemma:

Lemma 17. *The linear combination $\omega_s := (1-s)\omega_0 + s\omega_1$ is a folded symplectic form with fold Z.*

We begin by proving another auxiliary result.

Lemma 18. *Let μ be a closed 2-form on U. Then $p^*i^*\omega + t\mu$ is a folded symplectic form (on a possibly smaller tubular neighborhood of Z)if and only if $\mu(w, v)$ is nonvanishing on Z.*

Proof of Lemma 18. We must check that the top power of the form vanishes transversally on Z and that the pullback by the inclusion of Z is of maximal rank.

In order to have that $(p^*i^*\omega + t\mu)^n = (n-1)t(p^*i^*\omega)^{n-1} \wedge \mu + O(t^2)$ vanishes transversally at $t = 0$ we must have $(p^*i^*\omega)^{n-1} \wedge \mu$ is nonvanishing on Z. Note that the kernel of $(p^*i^*\omega)_z$ is spanned by w_z and v_z, so this happens if and only if $\mu(w, v)$ is nonvanishing on Z. The rank maximality condition is satisfied because $i^*(p^*i^*\omega + t\mu) = i^*\omega$. $\square$

Proof of Lemma 17. Let us see that both ω_0 and ω_1 are of the following form. We have $\omega_0 = p^*i^*\omega + t\mu_0$ where $\mu_0 = 2dtp^*\alpha + td(p^*\alpha)$ with $\mu_0(w, v) = 2$ on Z. As for ω_1 observe that $\iota_u(\omega - p^*i^*\omega) = 0$ for any vector field u in TZ and furthermore $\iota_w(\omega - p^*i^*\omega) = 0$ since its true for each term. Hence we have $\omega - p^*i^*\omega = 0$ on Z and consequently $\omega - p^*i^*\omega = t\mu_1$ for some $\mu_1 \in \Omega^2(U)$. Since ω is folded, we get for free that $\mu_1(w, v) = 0$ is nonvanishing on Z, and the choices made above guarantee that it is positive.

We can now write $\omega_s = p^*i^*\omega + t\mu_s$, where $\mu_s = (1-s)\mu_0 + s\mu_1$. Since $\mu_s(w, v)$ is positive on Z, the form ω_s is folded symplectic. □

We can now go back to the proposition we were proving. We were looking for a vector field v_s such that

$$\mathcal{L}_{v_s}\omega_s + \frac{d\omega_s}{ds} = 0.$$

This equation simplifies to

$$d\iota_{v_s}\omega_s = \omega_0 - \omega - 1.$$

Since $\omega_0 - \omega - 1$ is closed an vanishes on Z, which is a deformation retract of U, there exists a 1-form $\nu \in \Omega^1(U)$ that vanishes to second order on Z and such that $d\nu = \omega_0 - \omega - 1$. Then our equation is satisfied if

$$\iota_{v_s}\omega_s = \nu.$$

Because ω_s is a folded symplectic form, there exists a unique such vector field and vanishes to first order on Z. Integrating v_s we get an isotopy φ_s that satisfies $\frac{d\varphi_s}{ds} \circ \varphi_s^{-1} = v_s$ with $\varphi_0 = id$ and φ_s maps Z to Z. □

For Z not compact, replace $\epsilon \in \mathbb{R}^+$ by an appropriate continuous function $\epsilon : Z \to \mathbb{R}^+$ in the statement and proof of the proposition.

As a Corollary we obtain a local classification of folded symplectic manifolds up to folded symplectomorphism. It is the folded version of Darboux theorem in the symplectic case. This was originally proved by Martinet in [19].

Theorem 19. *Let (M, ω) be a $2n$-dimensional folded symplectic manifold and let z be a point in the folding hypersurface Z. Then there is a coordinate chart $(U; x_1, ..., x_n, y_1, ..., y_n)$ centered at z such that on U the set Z is given by $x_1 = 0$ and the folded symplectic form has the form*

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Proof. By the symplectic Darboux theorem, we have $i^*\omega = dx_2 \wedge dy_2 + ... + dx_n \wedge dy_n$. Now we apply Proposition 16 with $x_1 = t$ and $\alpha = \frac{1}{2}dy_1$. □

A particular example of folded symplectic manifolds, called origami manifolds, have been largely studied in [22]. This is the case where the line field V induced by the folded symplectic form on Z is a circle fibration instead of a general foliation.

Definition 30. *An origami manifold is a folded symplectic manifold (M,) whose nullfoliation on Z integrates to a principal S^1-fibration, called the nullfibration, over a compact base B.*

$$\begin{tikzcd} S^1 \arrow[r, hook] & Z \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

The form ω is called an origami form.

We assume that the S^1-action matches the induced orientation of the nullfoliation V. Observe also that if an origami manifold is folded symplectomorphic to another folded symplectic manifold, this last is also origami.

Example. Consider S^{2n} in $\mathbb{R}^{2n+1} \cong \mathbb{C}^n \times \mathbb{R}$ with coordinates $(x_1, y_1, ..., x_n, y_n, h)$ and let ω_0 be the restriction of the form $\sum_{i=1}^n dx_i \wedge dy_i$. It is a folded symplectic form in S^{2n} with folding hypersurface given by the intersection with the hyperplane $\{h = 0\}$, it is a $(2n-1)$-sphere. In polar coordinates the form we are restricting is written $\sum_{i=1}^n r_i dr_i \wedge d\theta_i$. Note that we have

$$\iota_{\frac{\partial}{\partial\theta_1}+...+\frac{\partial}{\partial\theta_n}}\omega_0 = -r_1 dr_1 - ... - r_n dr_n = hdh.$$

This vanishes on Z, and the nullfoliation is the Hopf fibration $S^1 \hookrightarrow S^{2n-1} \to \mathbb{C}P^{n-1}$. We deduce that (S^{2n}, ω_0) is an origami manifold. Indeed the $n = 1$ case is particularly simple: Z is S^1 itself.

5 Integrable systems in symplectic and Poisson manifolds

In this section we aim two different goals. First we will recall topological results developed in [8], the article that was developped using results in my own bachelor thesis [7]. We will then introduce for symplectic and Poisson geometry the notion of integrable system and present examples. In both contexts there is an action-angle coordinate theorem that we will state. The topological part of these theorems can be proved using a different perspective from the classical ones. We will detail this method, just as published in [8].

5.1 Generalizing Tischler theorem

Recall first a theorem in differential topology:

Theorem 20 (Tischler theorem)**.** *Let M^n be a compact manifold admitting a nowhere vanishing closed 1-form ω, then M^n is a fibration over S^1.*

We prove a generalization of Tichler's theorem that is only stated without proof for foliations without holonomy in [27]. We will need the following lemma to prove this result:

Lemma 21 (Ehresmann lemma [11])**.** *A smooth mapping $f : M^m \longrightarrow N^n$ between smooth manifolds M^m and N^n such that:*

1. *f is a surjective submersion, and*
2. *f is a proper map*

is a locally trivial fibration.

The theorem states:

Theorem 22. *Let M^n be a compact connected manifold endowed with k linearly independent closed 1-forms $\beta_i, i = 1, \ldots, k$ which are nowhere vanishing then M^n fibers over a torus $\mathbb{T}^k$.*

Proof. (of Theorem 22)We start by proving that the cohomology classes in $H^1(M^n, \mathbb{R})$, $\{[\beta_i]\}_{i=1}^k$ are all different. Assume the opposite β_i and β_j with $i \neq j$ such that $[\beta_i] = [\beta_j]$. Then there exists $f \in C^\infty(M^n)$ such that

$$\beta_i = \beta_j + df. \tag{2}$$

Since the 1-forms β_i are linearly independent the k-form $\beta_1 \wedge ... \wedge \beta_k$ is nowhere vanishing. Using equation 2 we obtain

$$\beta_i \wedge \beta_j = \beta_i \wedge (\beta_i + df) = \beta_i \wedge df. \tag{3}$$

But note that due to Weierstrass theorem f has a maximum and a minimum on a compact manifold, thus $\beta_i \wedge \beta_j$ vanishes at these points (where $df = 0$). This contradicts the fact that $\beta_1 \wedge ... \wedge \beta_k$ is nowhere vanishing.

Denote b_1 the first Betti number of M^n and θ the usual angular coordinate in S^1. It is well known that there exist b_1 maps $g_j : M^n \to S^1$ such that the set of 1-forms $g_j^*(d\theta)$ define a set of cohomology classes $[g_j^*(d\theta)]$ which is a basis of $H^1_{DR}(M^n, \mathbb{R})$. With this basis, we can express β_i as:

$$\beta_i = \sum_{j=1}^{b_1} a_{ij}\nu_j + dF_i, \text{ for } i = 1, ..., k.$$

Using the argument on Tischler theorem proof [27], we can choose appropriate $q_{ij} \in \mathbb{Q}$ $\forall i, j$, such that $\tilde{\beta_i} = \sum_{j=1}^p q_{ij}\nu_j + dF_i$ are still non-singular and independent. Taking suitable $N_i \in \mathbb{Z}$ we obtain forms $\beta_i' = N_i\tilde{\beta}_i$ such that

$$\beta_i' = \sum_{j=1}^{p} k_{ij}\nu_j + dH_i,$$

where $k_{ij} = N_i q_{ij} \in \mathbb{Z}$ and $H_i = N_i F_i \in C^\infty(M)$. Of course, they are also non singular and independent.

Without loss of generality we can assume $dH_i = 0$. Indeed, the image $H_i \in C^\infty(M^n)$ is contained in a closed interval because M^n is compact. Functions H_i quotients to S^1 with a projection π, and we can redefine $g_i := g_i + \pi \circ H_i$ for $i = 1, ..., k$.

Recall that the basis ν_j is defined as $\nu_j = g_j^*(d\theta) = d(\tilde{g_j})$, with $\tilde{g_j} = \theta \circ g_j$. Hence the forms β_i' can be written

$$\beta_i' = d(\sum_{j=1}^{p} p_{ij}\tilde{g_j}).$$

If we define the functions $\theta_i = \sum_{j=1}^p p_{ij}\tilde{g_j}$, then the induced mappings on the quotient $\tilde{\theta_i} : M^n \longrightarrow S^1$ are k submersions of M^n to S^1. Consider

$$\begin{aligned} \Theta : M^n &\longrightarrow S^1 \times ... \times S^1 = \mathbb{T}^k \\ p &\longmapsto (\tilde{\theta_1}(p), ..., \tilde{\theta_k}(p)). \end{aligned}$$

Since the forms β_i' are independent in $H^1(M^n, \mathbb{R})$ this implies $d\theta_i$ are independent seen as one-forms from M^n to $\mathbb{R}^k$ and so $d\tilde{\theta_i}$ are also independent into $\mathbb{T}^k$, this implies that Θ is a surjective submersion. Since M^n is compact, we can apply Ehresmann lemma (Lemma 21) and Θ defines a locally trivial fibration. $\square$

An easy non-trivial example of application of this theorem is the mapping torus.

Example. Let M be a manifold and f a diffeomorphism of M to itself. Take the cartesian product with $I \times I$ where $I$is the interval $[0, 1]$. Glue the boundary components by the homeomorphism:

$$M_f = \frac{M \times I \times I}{(x, 1, z) \sim (x, 0, z), (x, y, 0) \sim (x, y, 1)}.$$

Letting φ be a coordinate chart in M, then $(\varphi, \theta_1, \theta_2)$ are coordinates (not global, we need different neighborhood for the angles to be well defined) in M_f. Then the forms $d\theta_1$ and $d\theta_2$ are closed, nowhere vanishing and $d\theta_1 \wedge d\theta_2 \neq 0$. Applying Theorem 22, M_f fibers over $\mathbb{T}^2$.

Applying the theorem when $k = n$ we obtain the following as a corollary:

Corollary 23. *Let M^n be a compact connected manifold endowed with n linearly independent closed 1-forms $\beta_i, i = 1, \ldots, n$ which are nowhere vanishing then M^n is diffeomorphic to a torus $\mathbb{T}^n$.*

Proof. Applying Theorem 2.3, M^n fibers over a torus $\mathbb{T}^n$. From the invariance of domain theorem it is an immersion because the target space is n-dimensional too. Thus Θ defines a covering map but since M^n is connected it defines a diffeomorphism

$$M^n \cong \mathbb{T}^n.$$

$\square$

5.2 Integrable systems in symplectic manifolds

Let us recall the definition of integrable system in a symplectic manifold.

Definition 31. *An* ***integrable system*** *on a symplectic manifold (M^{2n}, ω) is a set of n functions $f_1, ..., f_n$ generically functionally independent (i.e. $df_1 \wedge ... \wedge df_n \neq 0$ on a dense set) and $\omega(X_{f_i}, X_{f_j}) = 0, \forall i, j$.*

We have that vector fields $X_{f_1}, ..., X_{f_n}$ are tangent to $F^{-1}(p)$. We can write then $T(F^{-1}(p))_p =< X_{f_1}, ..., X_{f_n} >_p$. As $\omega(X_{f_i}, X_{f_j}) = 0\ \forall i, j$ we deduce that ω vanishes in $L = F^{-1}(p)$. This leads to two interesting definitions.

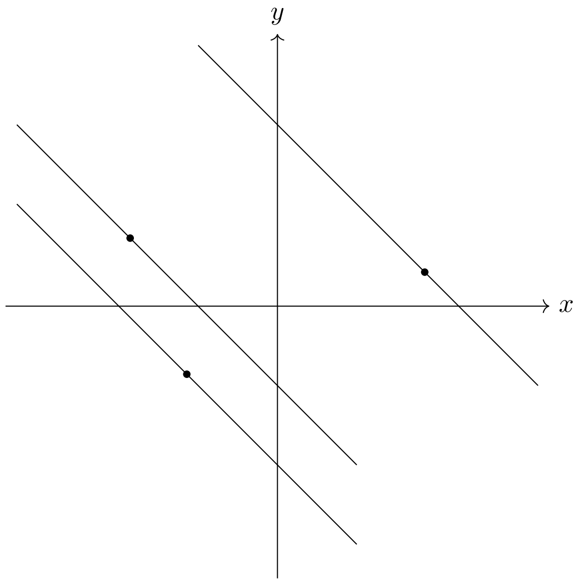
Definition 32. *A submanifold where the restriction of the symplectic form vanishes is called an* ***isotropic manifold****.*

Definition 33. *The particular case where the dimension of this submanifold is 1/2 dim(M) is called a* ***Lagrangian submanifold****. All the lagrangian submanifolds (the level sets) form a Lagrangian fibration.*

Example. Let's see a first very simple example of integrable system. Consider $(\mathbb{R}^2, \omega = dx \wedge dy)$ and $F = x + y$. Let's compute its associated vector field. A general vector field is of the form $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ so:

$$\begin{aligned} i_X\omega = dF &\implies \omega(X,\cdot) = d(x+y) \\ &\iff dx \wedge dy(X,\cdot) = dx + dy \\ &\iff ady - bdx = dx + dy \\ &\iff X = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}. \end{aligned}$$

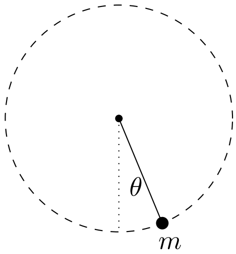
The lagrangian submanifolds are then generated by a point and the subspace $V =< (1,-1) >$. The fibration of $\mathbb{R}^2$ obtained by those lines, and a few fibres look like this.



It is clear that ω vanishes in these submanifolds: we only have the vector field X there, and we have that

$$\begin{aligned} \omega(X,X) &= dx \wedge dy(X,X) \\ &= dx(X)dy(X) - dy(X)dx(X) \\ &= -1 + 1 \\ &= 0. \end{aligned}$$

Example. An example of mechanical system which is also an integrable system is the simple pendulum. The manifold where the pendulum moves is S^1 and we can look its cotangent bundle as $T^*S^1 \cong [0, 2\pi]_\sim \times \mathbb{R}$ knowing that points at $(0,\xi)$ are identified with $(2\pi,\xi)$. We take the coordinates (θ,ξ) with θ the oriented angle between the rod and the vertical direction and ξ the velocity induced by θ.



To simplify, let's take the example where the mass and the length of the rod are 1. As we know, the Hamiltonian function for this system is

$$H(\theta, \xi) = \frac{\xi^2}{2} + 1 - \cos\theta.$$

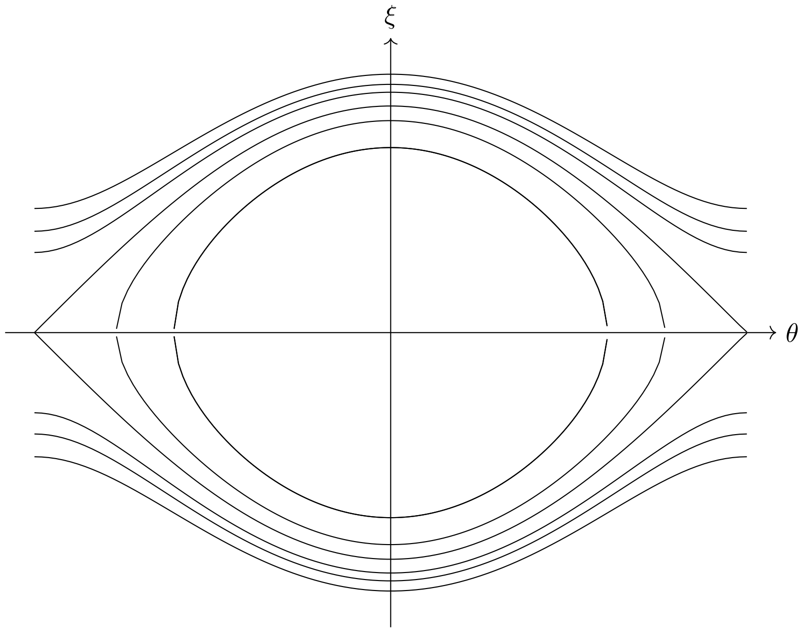
Let's compute the vector field associated to it. Since $dH = \xi d\xi + sin\theta d\theta$ we want:

$$\begin{aligned} i_X\omega = dH &\implies d\theta \wedge d\xi(a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\xi}, \cdot) = \xi d\xi + sin\theta d\theta \\ &\implies ad\xi - bd\theta = \xi d\xi + sin\theta d\theta. \end{aligned}$$

We deduce that

$$X_H = \xi\frac{\partial}{\partial\theta} - \sin\theta\frac{\partial}{\partial\xi}.$$

Some of the lagrangian fibres in the plane (θ, ξ) look like this.



As we can see, the lagrangian submanifolds are diffeomorphic to S^1, a 1-dimension torus. This is true only for regular values of the Hamiltonian, of course. If we consider the value 0, which is a singular point, the preimage is not an S^1 but a point.

Example (The 2-body problem [15])**.** The two-body problem is the system consisting of two bodies with masses m_1, m_2 and positions $q_1, q_2 \in \mathbb{R}^3$ moving under gravitational attraction. The equations of motion are deduced from the Newton laws:

$$m_i \ddot{q}_i = G m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3}, \quad i, j = 1, 2, \quad i \neq j,$$

where G is the gravitational constant. We can introduce the negative gravitational potential

$$U := m_1 m_2 \frac{G}{\|q_2 - q_1\|}.$$

So the equations are written

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i, j = 1, 2, \quad i \neq j.$$

We want to describe the equations of motion using the Hamiltonian formalism. The Hamiltonian function corresponds to the energy of the system and is obtained as the sum of kinetic and potential energy:

$$H(q_1, q_2, p_1, p_2) := E_{kin} - U = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - U,$$

where $p_i = m_i \dot{q}_i$ are the linear momenta. The evolution of the system is given by the Hamiltonian equations

$$\begin{cases} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i}. \end{cases}$$

And in our case $-\frac{\partial H}{\partial q_i} = \mathcal{G} m_1 m_2 \frac{q_j - q_i}{\|q_2 - q_1\|^3}$ Here the underlying symplectic structure is the canonical one for the cotangent space

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

From the equations of motion we observe that that

$$\dot{p}_1 + \dot{p}_2 = 0.$$

This means the quantity $p_1 + p_2$ is preserved. The center of mass moves with constant velocity and only the relative position $q := q_2 - q_1$ of the two bodies has to be solved from the equations. Let's introduce the following change of coordinates

$$\begin{aligned} g &= \nu_1 q_1 + \nu_2 q_2, & G &= p_1 + p_2, \\ q &= q_2 - q_1, & Q &= -\nu_2 p_1 + \nu_1 p_2, \end{aligned}$$

where $\nu_i = m_i/(m_1 + m_2)$. Note that g is the center of mass and G is the total linear momentum. The coordinate q is the relative position of the second body with respect to the first one. The other "momentum" coordinate Q is chosen such that the change of coordinates preserves the symplectic form (the change is "canonical"). This coordinates are called Jacobi coordinates.

In these coordinates the Hamiltonian is

$$H(g,q,G,Q) = \frac{\|G\|^2}{2\nu} + \frac{\|Q\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|q\|}$$

where $\nu = m_1 + m_2$ and $M = m_1 m_2/(m_1 + m_2)$.

Writing down the Hamiltonian equations explicitly

$$\begin{aligned} \dot{g} &= \frac{\partial H}{\partial G} = \frac{G}{\nu}, & \dot{G} &= -\frac{\partial H}{\partial g} = 0, \\ \dot{q} &= \frac{\partial H}{\partial Q} = \frac{Q}{M}, & \dot{Q} &= -\frac{\partial H}{\partial q} = -\frac{m_1 m_2 w}{\|q\|^3}, \end{aligned}$$

we see that total linear momentum G is preserved and that the center of mass moves with constant velocity $\frac{G}{\nu}$.

Physically this means that we are viewing the system as one body with coordinates q under the influence of the central force field of a body with mass M. Now we are facing a Hamiltonian system on $(\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^3$ with Hamiltonian function

$$H(q,Q) = \frac{\|Q\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|q\|}.$$

This is known as the Kepler problem.

The most important result about integrable systems in symplectic manifolds is the socalled Arnold-Liouville theorem [3]. This theorem describes semi-locally the integrable system around regular points.

Theorem 24 (Arnold-Liouville)**.** *Let (M^{2n}, ω) be a symplectic manifold and $F = (f_1, ..., f_n)$ an integrable system. Let p be a regular point (i.e. $df_1 \wedge ... \wedge df_n(p) \neq 0$). Note $F(p) = c$ and $F^{-1}(c) = L_c$ (fibre associated to c). Assuming L_c is compact and connected, then*

1. *$L_c \cong T^n$*
2. *the Liouville foliation is trivial in some neighborhood of the Liouville torus, that is, a neighborhood U of the torus L_c is the direct product of T^n and the disc D^n.*
3. *In a neighborhood of L_c, $U(L_c)$, there exist coordinates of the form $(\theta_1, ..., \theta_n, p_1, ..., p_n)$ and ω is written $\omega = \sum dp_i \wedge d\theta_i$. F only depends on $p_1, ..., p_n$.*

We will explain the non-classical proof of the first statement in [8].

Proof of 1 *in Arnold-Liouville.* Denote by L^n any connected component of $F^{-1}(c)$ (or all of it if assumed connected) and we also assume it is compact. Denote by X_i the Hamiltonian vector associated to f_i. Observe that

$$\begin{aligned} 0 &= \{f_i, f_j\} \\ &= \omega(X_i, X_j) \\ &= \iota_{X_i}\omega(X_j) \\ &= -df_i(X_j) = -X_j(f_i) \;\; \forall i,j = 1,...,n. \end{aligned}$$

and the vector fields $X_1, ..., X_n$ are tangent to L^n for all $p \in L^n$. So we can indeed write $T(L^n)_p = \langle X_{f_1}, ..., X_{f_n} \rangle_p$. Take now in $\mathbb{R}^n$ the canonical basis of vector fields $\{\partial_i = \frac{\partial}{\partial x_i}\}_{i=1}^n$ on $\mathbb{R}^n$ and consider their pullbacks by F, $S_i := F^*(\partial_i)$, which are vector fields in M^{2n} transverse to L^n. They satisfy:

$$S_i(f_j) = \delta_{ij}.$$

They are determined by this condition modulo T_pL^n.

Lemma 25. *Let $j : L^n \longrightarrow M^{2n}$ be the inclusion of the regular level set L^n into M^{2n}. Define the one-forms $\alpha_i = \iota_{S_i}\omega$. Then the one-forms $\beta_i = j^*\alpha_i$ are closed.*

Proof. By definition of S_i, we have $S_i(f_j) = \delta_{ij}$. Applying it $\forall i, j$:

$$\begin{aligned} \alpha_i(X_j) &= \omega(S_i, X_j) \\ &= -\omega(X_j, S_i) \\ &= -\iota_{X_j}\omega(S_i) \\ &= df_j(S_i) = S_i(f_j) = \delta_{ij}. \end{aligned}$$

To prove that β_i is closed, we just have to check that $d\alpha_i(X_i, X_j) = 0$ for all $X_i, X_j \in TL^n$.

$$\begin{aligned} d\alpha_i(X_j, X_k) &= X_k(\alpha_i(X_j)) - X_j(\alpha_i(X_k)) - \alpha_i([X_j, X_k]) \\ &= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \alpha_i(0) = 0. \end{aligned}$$

We conclude that $d(j^*\alpha_i) = d\beta_i = 0$ and so our forms β_i are closed in L^n. □

Lemma 26. *The 1-forms $\beta_1, ..., \beta_n$ are linearly independent and non-singular at all points of L^n.*

Proof. As seen in the previous lemma, we have that $\beta_i(X_j) = \delta_{ij}$. We deduce that $\beta_i = {X_i}^*$, by definition of dual basis. Since $X_1, ..., X_n$ form a basis of the tangent space at every point in L^n, we have that $\beta_1, ..., \beta_n$ form a basis of the cotangent space at every point in L^n. In particular all β_i are independent. This implies that $\beta_1 \wedge ... \wedge \beta_n$ is a volume form and so each of the forms is non-singular. □

The forms β_i in the preceding lemma are n closed 1-form which are nowhere vanishing and independent. Applying Corollary 23, $L^n \cong \mathbb{T}^n$. □

5.3 Integrable systems in Poisson manifolds

Heading back to the Poisson world, there exists also the notion of integrable system with similar results to the symplectic ones.

Definition 34. *An integrable system in a Poisson manifold (M, Π) of dimension $n = 2r + s$ with rank $\Pi = 2r$ is a s-tuplet of functions $F = (f_1, ..., f_s)$ such that*

1. *$f_1, ..., f_s$ are independent ($df_1 \wedge ... \wedge df_s \neq 0$ in an open dense set) and*
2. *the functions are in involution: $\{f_i, f_j\} = 0$ for all i,j.*

As in the symplectic case, we usually call $F = M \to \mathbb{R}^s$ the momentum (or moment) map. This function defines a foliation given by the fibres of F. We can denote this foliation as $\bar{\mathcal{F}}$. But there is another foliation that we can consider here. Recall that each of the functions f_i defines a Hamiltonian vector field defined as $X_{f_i} = \Pi(df_i, \cdot)$. If we consider the distribution generated by these vector fields $\mathcal{D} = \langle X_{f_1}, ..., X_{f_s} \rangle$. This distribution is involutive since we saw in Section 2.1 that $[X_{f_i}, X_{f_j}] = X_{\{f_i, f_j\}}$ which is 0 in this case.

A generic example would be the following.

Example. Consider the manifold $M = \mathbb{T}^r \times \mathbb{R}^s$ with coordinates

$$(x_1, \ldots, x_r, y_1, \ldots, y_s)$$

with the Poisson bivector field

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y}.$$

The functions $(y_1, ..., y_s)$ define an integrable system on (M, Π).

An action-angle coordinates theorems holds also in this general setting, proved in [18]. Denote the open subset where Π has rank $2r$ by M_r and the open set $df_1, ..., df_s$ are independent by U_F.

Theorem 27. *Let (M, Π) be a Poisson manifold of dimension n of maximal rank $2r$. Suppose that $F = (f_1, \ldots, f_s)$ is an integrable system on (M, Π), i.e., $r + s = n$ and the components of F are independent and in involution. Suppose that $m \in M$ is a point such that*

(1) $\mathrm{d}_m f_1 \wedge \ldots \wedge \mathrm{d}_m f_s \neq 0$*;*

(2) The rank of Π at m is $2r$;

(3) The integral manifold $\mathcal{F}_m$ of the distribution generated by $X_{f_1}, \ldots, X_{f_s}$, passing through m, is compact.

Then there exists $\mathbb{R}$-valued smooth functions $(\sigma_1, \ldots, \sigma_s)$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$ such that

1. *The manifold $\mathcal{F}_m$ is a torus $\mathbb{T}^r$.*

2. *The functions $(\theta_1, \ldots, \theta_r, \sigma_1, \ldots, \sigma_s)$ define an isomorphism $U \simeq \mathbb{T}^r \times B^s$;*

3. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial \sigma_i},$$

   in particular the functions $\sigma_{r+1}, \ldots, \sigma_s$ are Casimirs of Π (restricted to U);

4. *The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$, in particular, the functions $\sigma_1, \ldots, \sigma_s$ depend on the functions $f_1, \ldots, f_s$ only.*

The functions $\theta_1, \ldots, \theta_r$ are called angle coordinates, the functions $\sigma_1, \ldots, \sigma_r$ are called action coordinates and the remaining functions $\sigma_{r+1}, \ldots, \sigma_s$ are called transverse coordinates.

A more general definition are non-commutative integrable systems. Integrable systems are a particular case of this last ones. For the non-commutative case, an action-angle coordinate theorem holds too and the topological part can be proved using our methods presented before. We deduce the commutative case as a particular one. Recall the definition of a non-commutative integrable system in a Poisson manifold:

Definition 35. *Let (M, Π) be a Poisson manifold. An s-tuple of functions $F = (f_1, \ldots, f_s)$ on M is a* ***non-commutative (Liouville) integrable system*** *of rank $r \leq s$ on (M, Π) if*

1. *$f_1, \ldots, f_s$ are independent (i.e. their differentials are independent on a dense open subset of M) and the Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of M.*
2. *The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$ and $r + s = \dim M$.*

Let us introduce some notation. We denote the subset of M where the differentials $df_1, \ldots, df_s$ are independent by U_F and the subset of M where the vector fields $X_{f_1}, \ldots, X_{f_r}$ are independent by $M_{F,r}$.

On the open subset $M_{F,r} \cap U_F$ of M, the Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_r}$ define an involutive distribution of rank r. Let $\mathcal{F}$ be its foliation with, see [18]. When $\mathcal{F}_m$ is a compact r-dimensional manifold, the action-angle coordinate theorem proved in [18] states:

Theorem 28. *Let F be a non-commutative integrable system in (M, Π) of rank r, where $F = (f_1, \ldots, f_s)$ and suppose that $\mathcal{F}_m$ is compact, where $m \in M_{F,r} \cap U_F$. Then there exist $\mathbb{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbb{R}/\mathbb{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$, defined in a neighborhood U of $\mathcal{F}_m$, and functions $\phi_{kl} = -\phi_{lk}$, which are independent of $\theta_1, \ldots, \theta_r, p_1, \ldots, p_r$, such that*

1. *$\mathcal{F}_m$ is a torus $\mathbb{T}^r$.*
2. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbb{T}^r \times B^s$;*
3. *The Poisson structure can be written in terms of these coordinates as,*
$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{k,l=1}^{s-r} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$
4. *The leaves of the surjective submersion $F = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbb{T}^r \times B^s$ and as a consequence the functions $f_1, \ldots, f_s$ depend on $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ only.*

Let us now prove the first part of the theorem above using the alternative methods presented before.

Proof. (of 1.)

Consider $F = (f_1, \dots, f_s)$ the set of first integrals of the non-commutative integrable system. Consider the span of the Hamiltonian vector fields $X_i := \Pi(df_i, \cdot)$. From definition of the non-commutative integrable system at each point on the regular set, dimension of the vector space is r. Denote by α_i the 1-forms such that $\alpha_i(X_j) = \delta_{ij}$. They are not uniquely determined, but if we consider the inclusion j of the orbit into the manifold, then the 1-forms $\beta_i = j^*\alpha_i$ are uniquely determined. We can easily check that the forms β_i are closed:

$$\begin{aligned} d\beta_i(X_j, X_k) &= X_k(\beta_i(X_j)) - X_j(\beta_i(X_k)) - \beta_i([X_j, X_k]) \\ &= X_k(\delta_{ij}) - X_j(\delta_{ik}) - \beta_i(0) = 0 \end{aligned}$$

where in the last equality we have used that $[X_j, X_k] = X_{\{f_j, f_k\}}$ and from the definition of non-commutative integrable system $X_{\{f_j, f_k\}} = X_0 = 0$.

From the definition the dimension of the orbit is r and we have exactly r forms thus applying Corollary 23 we conclude that the orbit is a torus. From the regular value theorem, observe also that this orbit is the connected component through the point of the mapping given by $F = (f_1, \dots, f_s)$. $\square$

Finally, when $r = s$ we obtain as corollary the topological part of the action-angle theorem for commutative integrable systems.

Corollary 29. *Given an integrable system on a Poisson manifold $F = (f_1, \dots, f_s)$, the regular integral manifold $\mathcal{F}_m$ of the distribution generated by $X_{f_1}, \dots, X_{f_s}$, passing through m, is a torus of dimension r, $\mathbb{T}^r$.*

5.4 Integrable systems in b-symplectic manifolds

The next step is to introduce a new definition for integrable systems in b-symplectic manifolds. Indeed, b-Poisson manifolds are only a particular case of a Poisson manifold. However, if we apply results in Poisson Geometry, in the critical hypersurface we only obtain a distribution of rank $n-1$. Using the notions of b-functions and b-Hamiltonian vector fields, these results can be improved when restricted to the critical hypersurface. If we accept that the functions that define the integrable system can be b-functions, we can obtain a distribution of rank n in Z. The main issue is the following.

Let's say we have the b-symplectic form under a the desired action-angle coordinates expression:

$$\omega = \tfrac{c}{p_1} d\theta_1 \wedge dp_1 + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

It is clear that the vector fied $\frac{\partial}{\partial \theta_1}$ is not a Hamiltonian vector field for the function p_1. But following the definition of b-Hamiltonian vector field, it is actually the b-Hamiltonian vector field of $\log|p_1|$. This leads to the following definition.

Definition 36. *A b-integrable system on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is a set of b-functions $F = (f_1, \dots, f_n)$ that satisfy*

- *the functions are pairwise commuting, $\{f_i, f_j\} = 0$ for all i, j;*

- $df_1 \wedge \cdots \wedge df_n$ *is nonzero as a section of* $\Lambda^n({}^bT^*(M))$ *on a dense subset of* M *and on a dense subset of* Z.

After introducing this definition, a more accurate action-angle theorem was proved in [16].

Theorem 30 (Action-angle coordinates for b-integrable systems)**.** *Let* (M, ω) *be a b-symplectic manifold with critical hypersurface Z. Let F be a b-integrable system on (M, ω) and let $m \in Z$ be a regular point of the system lying inside the critical hypersurface. Assume that the integral manifold $\mathcal{F}_m$ containing m is compact, i.e. a Liouville torus. Then there exists an open neighborhood U of the torus $\mathcal{F}_m$ and a diffeomorphism*

$$(\theta_1, \dots, \theta_n, t, p_2, \dots, p_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function for Z, such that

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, the functions $t, p_2, \dots, p_n$ depend only on F. The number c is the modular period of the component of Z containing m.

The S^1-valued functions

$$\theta_1, \dots, \theta_r$$

are called angle coordinates *and the $\mathbb{R}$-valued functions*

$$t, p_2, \dots, p_r$$

are called action coordinates.

A very good source of integrable systems in b-symplectic manifolds is the b-cotangent lift. This is a similar procedure to the cotangent lift for integrable systems in symplectic manifolds, but using the b-cotangent bundle ${}^bT^*M$ instead of the usual cotangent bundle. This goes beyond the possibilities of this thesis but, as will explain in the last section: Conclusion, a similar construction of a cotangent bundle and cotangent lift could work for folded symplectic manifolds. This still has to be done and we may do it in the next months.

6 Integrable systems in folded-symplectic manifolds

We now reach the main goal of this thesis: define integrable systems in folded symplectic manifolds and obtain a similar result concerning action-angle coordinates.

6.1 Integrable systems in folded-symplectic manifolds

We start by defining a folded-symplectic integrable system.

Definition 37. *Let (M, ω) be a folded-symplectic manifold with folding hypersurface Z. Then an integrable system is $F = (f_1, ..., f_n)$, where the n functions f_i are such that in a dense set of Z they define independent Hamiltonian vector field. The functions are functionally independent on a dense set of M and their Hamiltonian vector fields commute with respect to ω everywhere.*

Observe that if the level set given by a regular value of the integrable system is assumed compact, we actually have a torus action as defined in the previous section. Let's analyse in depth this definition. At neighborhood outside Z the definition is the same as in the symplectic case, so nothing has to be said.

Consider p a point in Z, and applying folded-Darboux theorem the folded-symplectic form ω takes locally the form

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

The first remark that has to be done is that around this point **not every function f defines a Hamiltonian vector field**. This is indeed the fault of the singularity of ω. Let's see it for a simple example:

Example. Let $(U; x_1, y_1, ..., x_n, y_n)$ be a chart where ω takes the folded-Darboux form mentioned above. Take for example the function $f = x_1$. Let us try to compute a Hamiltonian vector field for this functions. Take

$$X = a_1 \frac{\partial}{\partial x_1} + b_1 \frac{\partial}{\partial y_1} + ... + a_n \frac{\partial}{\partial x_n} + b_n \frac{\partial}{\partial y_n}$$

a general vector field in U. Then

$$\begin{aligned} \iota_X \omega = -df &\implies -b_1 x_1 dx_1 = -dx_1 \\ &\implies b_1 = \frac{1}{x_1}. \end{aligned}$$

We obtain a non defined vector field: with this coordinates, x_1 is a defining function of Z and hence vanishes in Z. The coefficient b_1 is not well defined nor the vector field.

The problem comes from the singular part of the form. It is clear that the functions that that locally only depend in coordinates $(x_2, y_2, ..., x_n, y_n)$ will define without any problem Hamiltonian vector fields. But if we want the Hamiltonian vector field to have components in $\frac{\partial}{\partial x_1}$ or $\frac{\partial}{\partial y_1}$, more has to be said.

For a general function $f \in C^\infty(M)$, let us try to compute the Hamiltonian vector field. As before, denote a general vector field as

$$X = a_1 \frac{\partial}{\partial x_1} + b_1 \frac{\partial}{\partial y_1} + ... + a_n \frac{\partial}{\partial x_n} + b_n \frac{\partial}{\partial y_n}.$$

Imposing $\iota_X \omega = -df$, the components we are interested in are

$$\begin{aligned} a_1 &= -\frac{\partial f}{\partial y_1} \frac{1}{x_1} \\ b_1 &= \frac{\partial f}{\partial x_1} \frac{1}{x_1}. \end{aligned}$$

The only way that these components do not vanish and are defined is that $\frac{\partial f}{\partial x_1} = Cx_1$ for a constant C and that $\frac{\partial f}{\partial y_1} = 0$.

If we denote t the defining function of Z, we can now write locally around $p \in Z$ the set of functions as:

$$\begin{aligned} f_1 &= c_1 t^2 + h_1(x_2, y_2, ..., x_n, y_n) \\ f_2 &= c_2 t^2 + h_2(x_2, y_2, ..., x_n, y_n) \\ &\vdots \\ f_n &= c_n t^2 + h_n(x_2, y_2, ..., x_n, y_n), \end{aligned}$$

where $h_i C^\infty(M)$ Since the rank of the Hamiltonian vector fields associated to this functions has to be maximal in Z, we have that $c_i \neq 0$ for one of the functions, for instance assume $c_1 \neq 0$. Redefining

$$f_i := f_i - \tfrac{c_i}{c_1} f_1 \text{ for } i = 2, ..., n$$

then only the first functions has a component depending on the t variable. Finally, it is clear that $\frac{t^2}{2}$ is an integral: the Hamiltonian vector fields are all tangent to Z so t is constant along the orbits. In particular, we can take as first integral simply $f_1 := \frac{t^2}{2}$ instead of the one we had before. The geometric foliation given by the integrable system is not modified. This shows that we can assume that locally around Z the integral systems is of the form

$$F = (\tfrac{t^2}{2}, f_2, ..., f_n),$$

where $f_2, ..., f_n$ depend only in $x_2, ..., y_n$ for a Darboux local chart in a neighborhood of any point in Z.

Remark. Because of the discussion below, from now on during the rest of the thesis when speaking about an folded integrable system we will assume it has this form.

Let us show a basic example of a folded symplectic integrable system of this form.

Example. Consider $\mathbb{R}^n$ endowed with the canonical folded symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Let $F = (x_1^2/2, x_2, ..., x_n)$. If f_i are the components of F, then it is clear that

$$X_{f_i} = \frac{\partial}{\partial y_i}.$$

These n vector fields are independent everywhere, and they commute:

$$[X_{f_i}, X_{f_j}] = [\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}] = 0.$$

It defines an integrable system in $(\mathbb{R}^4, \omega)$.

7 Action-angle coordinates for integrable systems in folded-symplectic manifolds

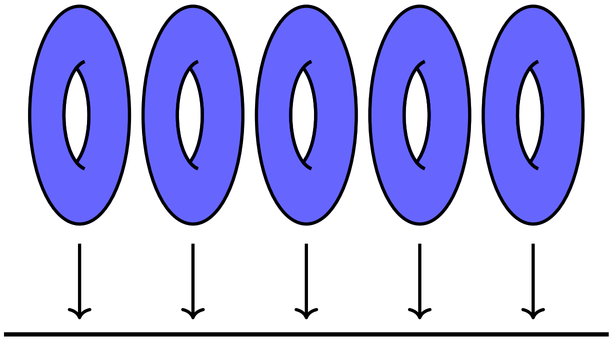
7.1 Liouville foliation and Carathéodory theorem

We first show that for a folded integrable system there is also a foliation by Liouville tori.

Proposition 31. *Let $p \in Z$ be a regular point of a folded-integrable system (M, ω, F). Assume that the integral manifold $\mathcal{F}_p$ is compact. Then there is neighborhood U of $\mathcal{F}_p$ and a diffeomorphism*

$$\varphi : U \cong \mathbb{T}^n \times B^n$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.



Fibration by Liouville tori

Proof. We use the same proof as in [18] and the only difference is for the first integral $f_1 = t^2/2$. The foliation given by the Hamiltonian vector fields of F is the same as the ones given by the level sets of $\bar{F} = (t, f_2, ..., f_n)$. This is clear since by definition the Hamiltonian vector field of $t^2/2$ is tangent to the level set of this function, and hence also to the level sets of t. Then choosing an arbitrary Riemannian metric on M, it defines a canonical projection $\psi : U \to \mathcal{F}_m$. Setting $\varphi := \psi \times \bar{F}$ we have a commuting diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "\bar{F}"'] & \mathbb{T}^n \times B^n \arrow[d, "\pi"] \\ & B^n \end{tikzcd}$$

which provides the necessary isomorphism. $\square$

Darboux-Carathodory theorems are Darboux-type theorems that keep track of an additional set of Poisson-commuting functions. They are useful in the study of integrable systems because integrable systems provide a maximal set of Poisson commuting functions.

Indeed Darboux-Carathodory theorem was used in [18] to prove an action-angle theorem in the general context of Poisson manifolds.

Theorem 32. *Let m be a point of a Poisson manifold (M, Π) of dimension n. Let $p_1, \ldots, p_r$ be r functions in involution, defined on a neighborhood of m, which vanish at m and whose Hamiltonian vector fields are linearly independent at m. There exist, on a neighborhood U of m, functions $q_1, \ldots, q_r, z_1, \ldots, z_{n-2r}$, such that*

1. *The n functions $(p_1, q_1, \ldots, p_r, q_r, z_1, \ldots, z_{n-2r})$ form a system of coordinates on U, centered at m;*

2. The Poisson structure Π *is given on* U *by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{4}$$

where each function $g_{ij}(z)$ *is a smooth function on* U *and is independent of* $p_1, \ldots, p_r, q_1, \ldots, q_r$.

The rank of Π *at* m *is* $2r$ *if and only if all the functions* $g_{ij}(z)$ *vanish for* $z = 0$.

In the case when the Poisson manifold is a symplectic manifold we obtain as a corollary the standard Darboux-Carathéodory theorem for symplectic manifolds.

Theorem 33 (Darboux-Carathéodory theorem for Symplectic manifolds)**.** *Let* m *be a point on a symplectic manifold* (M, ω). *Let* $f_1, ..., f_r$ *be* r *functions whose Hamiltonian vector fields are well defined and independent at* m. *Assume* r *functions are assumed to Poisson commute. Then in a neighborhood* U *of* m *there exists functions* $q_1, ..., q_r$ *and complementary coordinates* $x_1, y_1 ..., x_n, y_{n-r}$ *such that the set of functions* $(f_1, g_1, \ldots, f_n, g_n, x_1, y_1 \ldots, x_n, y_{n-r})$ *form a system of coordinates on* U, *centered at* m *and in these coordinates the symplectic form can be written*

$$\omega = \sum_{i=1}^{n} df_i \wedge dg_i + \sum_{i=1}^{i=n-r} dx_i \wedge dy_i$$

.

Through a combination of methods used in [18] to prove a Darboux-Carathéodory theorem in the Poisson setting and results in [6] we can prove a Darboux-Carathéodory theorem for folded symplectic manifolds.

This is going to be a key point in our proof of existence of action-angle coordinates.

Theorem 34 (Folded Darboux-Carathéodory)**.** *Let* $m \in Z$ *be a point of the folding hypersurface of a folded symplectic manifold* (M, ω). *Let* $f_1 = t^2/2, ..., f_r$ *be* r *functions whose Hamiltonian vector fields are well defined and independent at* m. *Function* t *is a locally defining function of* Z. *All* r *functions are assumed to Poisson commute. Then in a neighborhood* U *of* m *there exists functions* $q_1, ..., q_r$ *and complementary functions* $x_1, y_1, \ldots, x_{n-r}, y_{n-r}$ *such that*

1. *the* $2n$ *functions* $(t, f_2, ..., f_r, q_1, ..., q_r, x_1, y_1, \ldots, x_{n-r}y_{n-r})$ *form a system of coordinates on* U, *centered at* m;
2. *with these coordinates the folded symplectic form is written*

$$\omega = \sum_{i=1}^{r} dq_i \wedge df_i + \sum_{i=1}^{n-r} dx_i \wedge dy_i$$

.

Proof. Let p be a point in the folding hypersurface Z. By the folded Darboux theorem we know that locally in a neighborhood of p, ω has the form $\omega =$

$x_1dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i$. By the same discussion done when defining integrable systems on folded symplectic manifolds, in this neighborhood (or a smaller restriction if needed) we may assume that $f_2, ..., f_r$ do not depend on x_1, y_1. Consider $i : Z \hookrightarrow M$ and $i^*\omega$. We can restrict to $j : \{y_1 = 0\} \cap Z \hookrightarrow M$ too and we will have $i^* f_i = j^* f_i = f_i$ for $i = 2, ..., n$. We can apply now Darboux Carathéodory theorem for the symplectic case for the form $i^*\omega = j^*\omega$ and there exist $q_2, ..., q_r$ and $u_1, v_1, ..., u_{n-r}, v_{n-r}$ such that

$$i^*\omega = \sum_{i=2}^{r} dq_i \wedge df_i + \sum_{i=1}^{n-r} du_i \wedge dv_i.$$

Apply now Frobenius theorem to $X_{f_1}, ..., X_{f_r}$. There exist $(g_1, ..., g_r)$ with $X_{f_i} = \frac{\partial}{\partial g_i}$. Set $q_1 := g_1$. It satisfies $X_{q_1}(f_i) = -X_{f_i}(q_1) = -\delta_{i,1}$. In particular $dq_1(X_{f_1}) = X_{t^2/2}(q_1) = 1$.

Let us recall Proposition 16. Let v be an oriented non-vanishing section of V and $\alpha \in \Omega^1(M)$ a one-form such that $\alpha(v) = 1$. Then the following stands.

Proposition. *Assume Z is compact. Then there is a tubular neighborhood U of Z in M and an orientation preserving diffeomorphism $\varphi : Z \times (-\epsilon, \epsilon) \to U$ mapping $Z \times \{0\}$ onto Z such that*

$$\varphi^*\omega = p^* i^*\omega + d(t^2 p^*\alpha),$$

where $p : Z \times (-\epsilon, \epsilon) \to Z$ is the projection onto the first factor and t is the real coordinate in $(-\epsilon, \epsilon)$.

Rename now the variable u and v as x and y. Applying this last result with t (which is a defining function of Z) and $\alpha = \frac{1}{2}dq_1$ (which satisfies the hypothesis $\alpha(v) = 1$ taking $v = 2X_{f_1}$) we obtain that

$$\omega = \sum_{i=1}^{r} dq_i \wedge df_i + \sum_{i=1}^{n-r} dx_i \wedge dy_i.$$

□

7.2 Proof of action-angle coordinates

We proceed now with the statement of the proof of the action-angle theorem.

Theorem 35. *Let $F = (f_1 = t^2/2, ..., f_n)$ be a folded symplectic integrable system in (M, ω) and $p \in Z$ a regular point in the folding hypersurface. We assume the integral manifold $\mathcal{F}_p$ containing p is compact. Then there exist an open neighborhood U of the torus $\mathcal{F}_p$ and a diffeomorphism*

$$(\theta_1, ..., \theta_n, t, \sigma_2, ..., \sigma_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function of Z and such that

$$\omega_U = ctd\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, functions $t, p_2, ..., p_n$ depend only on F.

The S^1-valued functions

$$\theta_1, ..., \theta_n$$

are called angle coordinates and the $\mathbb{R}$-valued functions

$$t, p_2, ..., p_n$$

are called action coordinates.

Proof. We may assume that the integrable system is of the form $f_1 = t^2/2, f_2, ..., f_n$ as in the proposition. The vector field $X_{f_1}, ..., X_{f_n}$ define a torus action on each Liouville tori $\mathbb{T}^n \times \{b\}_{b \in B^n}$. But does it define a torus action in a neighborhood of the form $\mathbb{T}^n \times B^n$. We will use, as in [16], uniformization of periods.

We denote by φ_i^t the time-t-flow of the Hamiltonian vector fields X_{f_i}. Consider the joint flow of these Hamiltonian vector fields.

$$\begin{aligned} \varphi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) &\longrightarrow \mathbb{T}^n \times B^n \\ \big((t_1, \ldots, t_n), (x, y)\big) &\longmapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_n^{t_n}(x, y). \end{aligned}$$

The vector fields X_{f_i} are complete and commute with one another so this defines an $\mathbb{R}^n$-action on $\mathbb{T}^n \times B^n$. When restricted to a single orbit $\mathbb{T}^n \times \{b\}$ for some $b \in B^n$, the kernel of this action is a discrete subgroup of $\mathbb{R}^n$, a lattice Λ_b. We call Λ_b the period lattice of the orbit. The rank of Λ_b is n because the orbit is assumed compact.

The lattice Λ_b will in general depend on b. The idea of uniformization is to modify the action such that $\Lambda_b = \mathbb{Z}^n$ for all b. For any $b \in B^{n-1} \times \{0\}$ and any $a_i \in \mathbb{R}$ the vector field $\sum a_i X_{f_i}$ on $\mathbb{T}_n \times \{b\}$ is the Hamiltonian vector field of the function

$$a_i t^2/2 + \sum_{i=2}^{n} a_i f_i.$$

To perform the uniformization we pick smooth functions

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) : B^n \to \mathbb{R}^n$$

such that

1. $(\lambda_1(b), \lambda_2(b), \ldots, \lambda_n(b))$ is a basis for the period lattice Λ_b for all $b \in B^n$
2. λ_i^1 vanishes along $\{0\} \times B^{n-1}$ for $i > 1$. Here, λ_i^j denotes the j^{th} component of λ_i.

Such functions λ_i exist that satisfy the first condition (perhaps after shrinking B^n) by the implicit function theorem, using the fact that the Jacobian of the equation $\Phi(\lambda, m) = m$ is regular with respect to the s variables. We'll see now why the can be chosen to satisfy the second condition.

We define a uniformized flow using the functions λ_i as

$$\begin{aligned} \tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) &\to \mathbb{T}^n \times B^n \\ \big((s_1, \ldots, s_n), (x, b)\big) &\mapsto \Phi\left(\sum_{i=1}^{n} s_i \lambda_i(c), (x, b)\right). \end{aligned}$$

The period lattice of this $\mathbb{R}^n$ action is constant now, namely $\mathbb{Z}^n$, and hence the action naturally defines a $\mathbb{T}^n$ action.

We want to find now functions $\sigma_1, ..., \sigma_n$ such that their Hamiltonian vector fields are precisely the ones constructed above $Y_i = \sum_{j=1}^n \lambda_i^j X_{f_j}$. Denote from now on λ_1^1 as c.

Compute now the Lie derivative of these vector fields, using Cartan's formula:

$$\begin{aligned}\mathcal{L}_{Y_i}\omega &= d\iota_{Y_i}\omega + \iota_{Y_i}d\omega \\ &= d(-\sum_{j=1}^n \lambda_i^j df_j) \\ &= -\sum_{j=1}^n d\lambda_i^j \wedge df_j\end{aligned}$$

We deduce

$$\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = \mathcal{L}_{Y_i}(-\sum_{j=1}^n d\lambda_i^j \wedge df_j).$$

In the last equality we used the fact that λ_i^j are constant on the level sets of F. In [18] it is shown that given a complete vector field Y of period 1 and a bivector field P such that $\mathcal{L}_Y\mathcal{L}_Y P = 0$ then $\mathcal{L}_Y P = 0$. If instead of a bivector field we take a 2-form, the same proof works using the pullback of the flow of Y instead of pushforward.:

Lemma 36. *If Y is a complete vector field of period 1 and ω is a 2-form such that $\mathcal{L}_Y\mathcal{L}_Y\omega = 0$ then $\mathcal{L}_Y\omega = 0$.*

Proof. Denote $v = \mathcal{L}_Y\omega$. Denote ϕ_t the flow of Y. For any point p we have

$$\begin{aligned}\frac{d}{dt}\left(\phi_t^*\omega_{\phi_{-t}(p)}\right) &= (\phi_t)^*(\mathcal{L}_Y\omega_{\phi_{-t}(p)}) \\ &= {\phi_t}^* v_{\phi_{-t}(p)} \\ &= v_p\end{aligned}$$

In the last equality we used that $\mathcal{L}_Y v = 0$. Integrating we obtain

$$(\phi_t)^*\omega_{\phi_{-t}(p)} = \omega_p + tv_p.$$

At time $t = 1$ the flow is the identity because Y has period 1 and hence $v_p = 0$. $\square$

Applying this lemma to the vector fields Y_i, we deduce that they preserve the folded-symplectic structure i.e. $\mathcal{L}_{Y_i}\omega = 0$.

It is required now to prove that each $\iota_{Y_i}\omega$ has a primitive i.e. is locally an exact form. Write ω as an exact form locally around our level set L.

Lemma 37. *In a neighborhood $U(L)$ of the level set, the folded symplectic form can be written*

$$\omega = d\alpha.$$

This lemma is a consequence of the following result in [28].

Theorem 38 (Relative Poincaré lemma)**.** *Let $N \subset M$ a closed submanifold and $i : N \hookrightarrow M$ the inclusion map. Let ω a closed k-form on M such that $i^*\omega = 0$. Then there is a $k-1$-form α on a neighborhood of N in M such that $\omega = d\alpha$.*

Of course, using the condition that the Hamiltonian vector fields commute with respect to ω we have that the hypothesis holds in our context. An interesting fact is that α can be taken invariant with respect to the toric action, which is stated:

Lemma 39. *If $\omega = d\alpha$ in $U(L)$ then we can find an $\bar{\alpha}$ such that for any X_i fundamental vector field of the torus action we have*

$$\mathcal{L}_{X_i}\bar{\alpha} = 0.$$

Proof. Define the new $\bar{\alpha}$ as

$$\bar{\alpha} = \int_{\mathbb{T}^n} \varphi^* \alpha d\mu,$$

which is clearly invariant by definition. Now since $\mathcal{L}_{X_i}\omega = 0$, the form ω is invariant and:

$$\begin{aligned} \omega &= \int_{\mathbb{T}^n} \omega \\ &= \int_{\mathbb{T}^n} d\alpha \\ \implies \omega &= d(\int_{\mathbb{T}^n} \alpha). \end{aligned}$$

□

We have now an α such that locally in $U(L)$ the folded symplectic form is written $\omega = d\alpha$ and α is invariant by the action. This means that $\mathcal{L}_{Y_i}\alpha = 0$ for all fundamental vector field of the action. Then applying Cartan's formula

$$\begin{aligned} \iota_{Y_i}\omega &= \iota_{Y_i} d\alpha \\ &= d\iota_{Y_i}\alpha. \end{aligned}$$

We deduce that the fundamental vector fields are Hamiltonian. Denoting $\sigma_1, ..., \sigma_n$ these Hamiltonian functions, we have now candidates for that "action" coordinates. Observe that $\sigma_1 = ct^2$ since $\lambda_i^1 = 0$ for all $i < n$.

By the Darboux-Carathéodory theorem we have that there exists a coordinate system

$$(\sigma_1, ..., \sigma_n, q_1, ..., q_n)$$

such that

$$\omega = d\sigma_1 \wedge dq_1 + \sum_{i=2}^{n} d\sigma_i \wedge dq_i.$$

Since the vector fields X_{σ_i} are Hamiltonian fundamental vector fields of the $\mathbb{T}^n$-action, in the local chart the flow of the vector fields gives a linear action on the q_i coordinates. This implies that the functions $\sigma_1 = ct^2/2, ..., \sigma_n$ that were defined in an open set U of the point can be extended to the whole set $U' = \sigma^{-1}(\sigma(U))$. Denote the same way these extensions. The Hamiltonian vector fields of σ_i

had period one, so functions q_i can be viewed as angle variables θ_i. It only remains to check if the extended functions define a system of coordinates in the neighborhood of the torus and that ω has the desired form.

Observe first that obviously $\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\theta_i}) = \delta_{ij}$ by the own definition of θ_i. In the original neighborhood U we had that $\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\sigma_j}) = \omega(X_{\theta_i}, X_{\theta_j}) = 0$. Applying the definition of exterior derivative, using that ω is closed and that he vector fields commute we obtain:

$$\begin{aligned} d\omega(X_{\theta_i}, X_{\theta_j}, X_{\sigma_k}) &= X_{\theta_i}(\omega(X_{\theta_j}, X_{\sigma_k})) - X_{\theta_j}(\omega(X_{\theta_i}, X_{\sigma_k})) \\ &+ X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j})) \\ &= 0 \end{aligned}$$

Using that $\omega(X_{\sigma_i}, X_{\theta_j}) = \delta_{ij}$ for all i and j, we obtain

$$X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j})) = 0.$$

In particular, flowing the vector fields X_{σ_k} we obtain that the relation $\omega(\frac{\partial}{\partial\sigma_i}, \frac{\partial}{\partial\sigma_j}) = 0$ holds in the whole neighborhood U'. We conclude that ω has the desired form

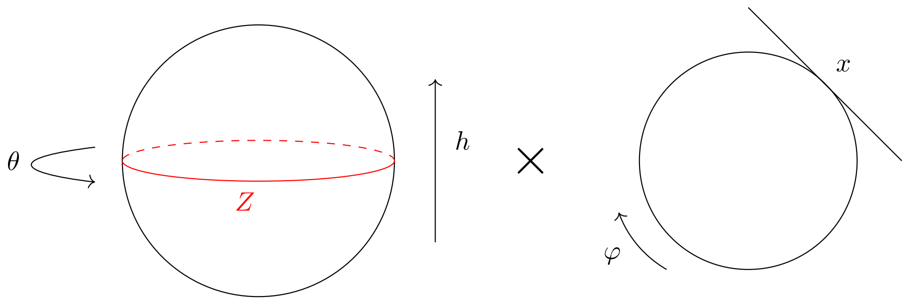
$$\omega = ctdt \wedge d\theta_1 + \sum_{i=2}^{n} d\sigma_i \wedge d\theta_i.$$

In particular the derivatives of the functions $\sigma_1, ..., \sigma_n, \theta_1, ..., \theta_n$ are independent on U and hence define a coordinate system. Taking for instance $p_i := -\sigma_i$ then the form in the neighborhood of the torus is written

$$\omega = ctd\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

This proves the action-angle coordinate theorem. $\square$

Example. Consider the manifold $M = S^2 \times T^*S^1$ with coordinates (h, θ, φ, p), endowed with the symplectic form $\omega = hdh \wedge +dp \wedge d\varphi$.



Take for integrals the functions $f_1 = h^2/2$ and $f_2 = p$. Their associated Hamiltonian vector fields are

$$X_{f_1} = \frac{\partial}{\partial\theta} \text{ and } X_{f_2} = \frac{\partial}{\partial\varphi}.$$

This vector fields are independent everywhere and commute. At any point of the hypersurface $m = (0, \theta_0, \varphi_0, x_0)$, the orbit is given by the set $\mathcal{F}_m = \{(0, \theta, \varphi, x_0) |\ \theta, \varphi \in [0, 2\pi]\} \cong \mathbb{T}^2$.

8 Relation with b-symplectic manifolds : Deblogging

The goal of this section is to use a desingularizing tool called "Deblogging" to obtain, with a different method, action-angle coordinates for the desingularization of a b-integrable system. As described in [10], when desingularizing certain b^m structures, in particular the odd ones, we obtain folded-symplectic structures.

8.1 Symplectic b^m-manifolds

One of the research directions has been to generalize b-structures and consider more degenerate singularities of the Poisson structure. This is the case of b^m-Poisson structures, for which ω^n has a singularity of A_n-type in Arnolds list of simple singularities [1] [2]. It is convenient, as in the b case, to consider the dual approach and work with forms for their study.

Definition 38. *A symplectic b^m-manifold is a pair (M^{2n}, Z) with a closed b^m-two form ω which has maximal rank at every $p \in M$.*

Such as in the b-symplectic case, there exists a b^m-Darboux theorem proved in [10]

Theorem 40 (b^m-Darboux theorem, [10])**.** *Let ω be a b^m-symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a coordinate chart $(x_1, y_1, ..., x_n, y_n)$ centered at p such that the hypersurface Z is locally defined by $\{y_1 = 0\}$ and*

$$\omega = dx_1 \wedge \tfrac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Dualizing we obtain the Darboux form for the b^m-Poisson bivector field,

$$\Pi = y_1^m \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}.$$

To describe this process of deblogging we need a few definitions and properties from [24].

Definition 39. *A Laurent Series of a closed b^m-form ω is a decomposition of ω in a tubular neighborhood U of Z of the form*

$$\omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\hat{\alpha}_i)x^i) + \beta,$$

where $\pi : U \to Z$ is the projection, where each $\hat{\alpha}_i$ is a closed form on Z, and β is form on U.

And there is a result concerning this decomposition of ω.

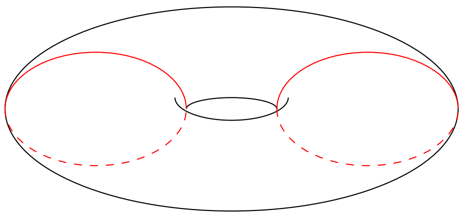
Proposition 41. *In a tubular neighborhood of Z, every closed b^m-form ω can be written in a Laurent form and the restriction of $\sum_{i=0}^{m-1} \pi^*(\hat{\alpha}_i)x^i$ and β to Z are well-defined closed 1 and 2-forms respectively.*

Some easy examples of b^m-surfaces are detailed in [23].

Examples.

- Consider the sphere S^2. A b^m-symplectic form in it is $\omega = \frac{1}{h^m} dh \wedge d\theta$, where h is the height and θ the angle. The critical set in this case is the equator.

- Let $\mathbb{T}^2$ the torus as a quotient of the plane: $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Consider in it the form
$$\omega = \frac{1}{(\sin 2\pi y)^n} dx \wedge dy.$$
It's a b^m-symplectic structure on $\mathbb{R}^2$. Since the action of $\mathbb{Z}$ leave the form invariant, the form descends to the quotient. The critical set $Z = \{y \in \{0, \frac{1}{2}\}$ is more sophisticated: it has two connected components.



- Taking the b^m-symplectic structure in the sphere S^2, it is invariant under the antipodal action if $m = 2k + 1$ is even. Then it yields a folded symplectic form in $\mathbb{RP}^2$, and the critical set is the equator modulo identifying the antipodal points.

8.2 Desingularizing b^m-symplectic structures

We will here explain briefly how b^m-symplectic structures can be desingularized. The parity of m give rise to two different results. We only detail the odd case since the obtained 2-form is a folded-symplectic form, and we only state the result for the even case.

Consider a manifold M with a b^{2k+1}-symplectic structure given by a b^{2k+1}-symplectic form ω. Writing $m = 2k + 1$, in a tubular neighborhood U of Z we know by the previous subsection that ω takes the form

$$\omega = \frac{dx}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\hat{\alpha}_i)x^i) + \beta$$

where $\pi : U \to Z$ is the projection, where each $\hat{\alpha}_i$ is a closed form on Z, and β is form on U.

Let a function $f \in C^\infty(\mathbb{R})$ satisfying

- $f(x) = f(-x)$
- $f'(x) > 0$ if $x > 0$
- $f(x) = x^2 - 2$ if $x \in [-1, 1]$

- $f(x) = \log(|x|)$ if $k = 0$, $x \in \mathbb{R} \setminus [-2, 2]$
- $f(x) = -\frac{1}{(2k+2)x^{2k+2}}$ if $k > 0$, $x \in \mathbb{R} \setminus [-2, 2]$.

Then define

$$f_\epsilon(x) := \frac{1}{\epsilon^{2k}} f\left(\frac{x}{\epsilon}\right)$$

We consider now the 2-form

$$\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k} \pi^*(\alpha_i) x^i) + \beta.$$

For this form, the following hold.

Theorem 42. *The 2-form ω_ϵ is a folded symplectic form which coincides with ω outside an ϵ-neighborhood of Z.*

Which leads to

Theorem 43. *A manifold admitting a b^{2k+1}-symplectic structure also admits a folded symplectic structure.*

This is a powerful tool that can also be applied to integrable systems.

When m is even, the desingularized form is a symplectic form.

Theorem 44. *A manifold admitting a b^{2k}-symplectic structure also admits a symplectic structure.*

In particular the topological constraints that apply for symplectic structures also apply for b^{2k}-symplectic structures.

8.3 Desingularizing integrable systems

Consider now $F = (f_1, ..., f_n)$ a b-integrable system. At any point in the critical hypersurface Z, we can apply the action-angle coordinate theorem for b-integrable systems obtaining coordinates $(\theta_1, \ldots, \theta_n, t, p_2, \ldots, p_n) : U \to \mathbb{T}^n \times B^n$ where t is a defining function for Z and the form is written

$$\omega|_U = \frac{c}{t} d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

Moreover, the functions $t, p_2, \ldots, p_n$ depend only on F. The Hamiltonian vector fields are indeed spanned by $\langle \frac{\partial}{\partial \theta_i} \rangle$. If we desingularize the form via the methods detailed in the previous section, we obtain the folded symplectic form

$$\omega_\epsilon = \frac{C}{\epsilon^2} t d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

The nice thing is that now the b-Hamiltonian vector fields that we had before, which were spanned by $\langle \frac{\partial}{\partial \theta_i} \rangle$ are now trivially independent Hamiltonian vector fields for our folded symplectic structure. Hence the action coordinates define an integrable system.

Theorem 45. *Let $F = (f_1 = c\log t, ...f_n)$ be an integrable system in a b-symplectic manifold (M, ω), where t is a defining function of the critical hypersurface Z. Let $F = (t^2/2, p_2, ..., p_n)$, where $(t, p_2, ..., p_n)$ are the action functions obtained by the action-angle theorem in b-symplectic manifolds. Then F defines a folded integrable system in the manifold M equipped with the desingularized form*

$$\omega_\epsilon = \frac{C}{\epsilon^2} t d\theta_1 \wedge dt + \sum_{i=2}^{n} d\theta_i \wedge dp_i \text{ for any small enough } \epsilon > 0,$$

where C is a constant.

9 Celestial Mechanics

We may now present examples of folded symplectic structures that appear in Celestial Mechanics problems, as detailed in [9]. We then do a similar analysis of the geometric structure for a change of variables in [17], which is done for the first time. We use this geometric structure to prove in a faster way a theorem in the same paper.

9.1 Double collision in three body problem

Recall the Kepler problem introduced in section 5.2. The Hamiltonian describing the system, in Jacobi coordinates, is:

$$H(q, Q) = \frac{\|Q\|^2}{2M} - \mathcal{G}\frac{m_1 m_2}{\|q\|}.$$

This is defined for $(q, Q) \in (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}^2$. The second-order differential equation associated to it is

$$\ddot{q} = -\mathcal{G}\frac{(m_1 + m_2)q}{\|q\|^3} \quad q \in \mathbb{R}^2.$$

Studying this equation in three dimensions instead of two can be interesting for studying binary collisions in the three-body problem. This problem cannot be restricted to two dimensions in general. To regularize this problem we will use the quaternion algebra $\mathbb{U}$. Recall that they consist of elements of the form

$$u = u_0 + iu_1 + ju_2 + ku_3,$$

where the elements i, j, k satisfy all the following identities.

- $i^2 = -1$
- $j^2 = -1$
- $k^2 = -1$
- $ij = -ji = k$
- $jk = -kj = i$
- $ki = -ik = j$.

We identify $\mathbb{U}$ with $\mathbb{R}^4$, using the vector $u = (u_0, u_1, u_2, u_3)$. Denote now u^* when speaking about the quaternion element. Define the mapping

$$u \mapsto \frac{uu^*}{2}.$$

We obtain as image of this mapping the set of quaternions that have a vanishing k component. Hence it can be identified with $\mathbb{R}^3$. For a given number v, the preimage is given by a one-parameter family of the form $l\dot{e}^{k\theta} = \dot{l}(\cos\theta + k\sin\theta)$, for $\theta \in S^1$. If we write the map explicitly we have:

$$\begin{aligned} v_0 &= \frac{u_0^2 - u_1^2 - u_2^2 + u_3^2}{2} \\ v_1 &= u_0u_1 - u_2u_3 \\ v_2 &= u_0u_2 + u_1u_3. \end{aligned}$$

This transformation is known as the Kustaanheimo-Stiefel transformation. Choosing a solution with $u_3 = 0$ and denoting the space with coordinates (v_0, v_1, v_2) with momenta (V_1, V_2, V_3) the symplectic form is written:

$$\begin{aligned} \omega &= dv_0 \wedge dV_0 + dv_1 \wedge dV_1 + dv_2 \wedge dV_2 \\ &= (u_0du_0 - u_1du_1 - u_2du_2) \wedge dV_0 + (u_0du_1 + u_1du_0) \wedge dV_1 + (u_0du_2 + u_2du_0) \wedge dV_2. \end{aligned}$$

Computing the top wedge $\omega^3 = (u_0^3 - u_1^2u_0 - u_2^2u_0)du_0 \wedge dV_0 \wedge du_1 \wedge dV_1 \wedge du_2 \wedge dV_2$. The coefficient is indeed $2u_0v_0$. This is a hyperbolic-like m-folded symplectic structure.

9.2 Total Collapse in the three body problem

Consider bodies with masses m_1, m_2, m_3 and positions $\mathbf{q_1} = (q_1, q_2, q_3)$, $\mathbf{q_2} = (q_4, q_5, q_6)$ and $\mathbf{q_3} = (q_7, q_8, q_9)$. We denote its momenta the same way p_1,...,p_9. Consider also the 9×9 matrix $M := diag(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$.

The three body problem is governed by the Hamiltonian function

$$H(p, q) = \tfrac{1}{2} p^T M^{-1} p - U(q),$$

where

$$U(q) = \frac{m_1m_2}{|q_1 - q_2|} + \frac{m_1m_3}{|q_1 - q_3|} + \frac{m_2m_3}{|q_2 - q_3|}.$$

We will assume that the center of mass remains at the origin, i.e.

$$m_1\mathbf{q_1} + m_2\mathbf{q_2} + m_3\mathbf{q_3} = 0.$$

Following [20] we introduce

$$r := \sqrt{q^T M q}, \quad s := \tfrac{q}{r}, \quad z := p\sqrt{r}.$$

Note that $r = 0$ corresponds to triple collisions. We then find

$$\begin{aligned} \dot{r} &= \sqrt{r}\langle s, z\rangle \\ \dot{s} &= r^{-3/2}(M^{-1}z - \langle s, z\rangle s) \\ \dot{z} &= r^{-3/2}(\nabla U(s) + \tfrac{1}{2}\langle s, z\rangle z). \end{aligned}$$

Denoting $\nu = \langle s, z\rangle$ and multiplying the vector field by $r^{3/2}$ we obtain

$$\begin{aligned} r' &= \nu r \\ s' &= M^{-1}z - \nu s) \\ z' &= \nabla U(s) + \tfrac{1}{2}\nu z. \end{aligned}$$

Denote $(\bar m_1, ..., \bar m_9) := (m_1, m_1, m_1, m_2, m_2, m_2, m_2, m_3, m_3, m_3)$. Then the inverse of the chart is

$$\begin{aligned} q_i &= r s_i, \;\; i = 1, .., 8 \\ q_9 &= r\sqrt{\frac{1 - \sum_{i=1}^{8} s_i^2 \bar m_i}{\bar m_9}} \\ p_i &= \frac{z_i}{\sqrt{r}}. \end{aligned}$$

With this new coordinates, we can now compute the standard symplectic form $\omega = \sum_{i=1}^{9} dq_i \wedge dp_i$. One by one the new derivatives are

$$\begin{aligned} dq_i &= s_i dr + r ds_i \;\; i = 1, ..., 8 \\ dq_9 &= \sqrt{\frac{1 - \sum_{i=1}^{8} s_i^2 \bar m_i}{\bar m_9}} dr - \sum_{i=1}^{8} \frac{s_i \bar m_i r}{\sqrt{\bar m_9 (1 - \sum_{i=1}^{8} s_i^2 \bar m_i)}} ds_i \\ dp_i &= -\tfrac{1}{2} z_i r^{-3/2} dr + \tfrac{1}{\sqrt{r}} dz_i \end{aligned}$$

Denote $\mu = (1 - \sum_{i=1}^{8} s_i^2 \bar m_i)$ and we compute now the symplectic form in the new coordinates:

$$\begin{aligned} \sum_{i=1}^{9} dq_i \wedge dp_i &= \sum_{i=1}^{8} (s_i dr + r ds_i) \wedge (-\tfrac{1}{2} z_i r^{-3/2} dr + \tfrac{1}{\sqrt{r}} dz_i) \\ &\quad + (\sqrt{\frac{\mu}{\bar m_9}} dr - \sum_{i=1}^{8} \frac{s_i \bar m_i r}{\sqrt{\bar m_9 \mu}} ds_i) \wedge (-\tfrac{1}{2} z_9 r^{-3/2} dr + \tfrac{1}{\sqrt{r}} dz_9) \\ &= \sum_{i=1}^{8} (\frac{s_i}{\sqrt{r}} dr \wedge dz_i + \frac{z_i}{2\sqrt{r}} dr \wedge ds_i + \sqrt{r} ds_i \wedge z_i) \\ &\quad + \sqrt{\tfrac{\mu}{\bar m_9 r}} dr \wedge dz_9 - \sum_{i=1}^{8} \frac{s_i \bar m_i r}{\sqrt{\bar m_9 \mu r}} ds_i \wedge dz_9 - \sum_{i=1}^{8} \frac{s_i \bar m_i}{2\sqrt{\bar m_9 \mu r}} z_9 dr \wedge ds_i. \end{aligned}$$

To compute the maximum wedge of ω, we can just look at the terms that will remain i.e. the term for the form $ds_1 \wedge dz_1 \wedge ... \wedge dr \wedge dz_9$. The only way to obtain it is combining the terms with $ds_i \wedge dz_i$ with the $dr \wedge dz_9$ one. We obtain

$$\omega^n = \sqrt{\frac{\mu r^7}{\bar m_9}} ds_1 \wedge dz_1 \wedge ... \wedge dr \wedge dz_9.$$

This is a $\frac{7}{2}$-folded symplectic structure.

9.3 Restricted three body problem

Following the same idea of the previous section, we can now study how the geometric structure is changed in the study of the restricted three body problem done in [17]. This is going to be done for the first time.

We consider the circular planar restricted three-body problem in a rotating coordinate system $q = (q_1, q-2)$ of rotational frequency equal to 1. There is a larger primary m_1 of mass $1-\mu$ at the origin and the smaller primary one m_2 of mass μ at the position $(-1, 0) = e_2$, with $\mu \in [0, \frac{1}{2}]$. Let $p = (p_1, p_2)$ be the momentum variables conjugated to q and then the motion of the zero mass particle m_3 is given by

$$H = \frac{|p|^2}{2} + q_2 p_1 - q_1 p_2 - \frac{1}{|q|} + \mu \left(\frac{1}{|q|} - \frac{1}{|q-e_2|} - p_2 \right).$$

The binary collision of the third body with the primary mass can be regularized by using McGehee coordinates. For the restricted problem these coordinates do not work, but the same ideas can be applied.

Let us start introducing the usual canonical transformation to polar coordinates.

$$\begin{aligned} q_1 &= Q_1 \cos Q_2 \\ p_1 &= P_1 \cos Q_2 - \frac{P_2}{Q_1} \sin Q_2 \\ q_2 &= Q_1 \sin Q_2 \\ p_2 &= P_1 \sin Q_2 + \frac{P_2}{Q_1} \cos Q_2. \end{aligned}$$

The coordinate $Q_1 = r$ is the radial one: the distance between the larger body at the origin and the third one m_3. The angular coordinate is $Q_2 = \theta$, the angle between the q_1-axis and the radius vector. The standard symplectic form here is $\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. We apply the change to the symplectic form.

$$\begin{aligned} dq_1 \wedge dp_1 + dq_2 \wedge dp_2 &= d(Q_1 \cos Q_2) \wedge d(P_1 \cos Q_2 - \tfrac{P_2}{Q_1} \sin Q_2) \\ &+ d(Q_1 \sin Q_2) \wedge d(P_1 \sin Q_2 + \tfrac{P_2}{Q_1} \cos Q_2) \\ &= 0 dQ_1 \wedge dP_1 + (\cos {Q_2}^2 + \sin {Q_2}^2) dQ_1 \wedge dP_1 + 0 dQ_1 \wedge dP2 \\ &+ 0 dQ_2 \wedge dP_1 + (\cos {Q_2}^2 + \sin {Q_2}^2) dQ_2 \wedge dP_2 \\ &= dQ_1 \wedge dP_1 + dQ_2 \wedge dP_2 \end{aligned}$$

With these coordinates the Hamiltonian is written:

$$H = \frac{P_1^2 + \frac{P_2^2}{Q_1^2}}{2} - P_2 - Q_1^2 + \mu \left(\frac{1}{Q_1} - (Q_1^2 + 1 + 2Q_1 \cos Q_2)^{-1/2} - P_1 \sin Q_1 - \frac{P_2}{Q_1} \cos Q_2 \right)$$

Denoting $Q_1 = r$ and $Q_2 = \theta$, we introduce the components of velocity $x = \dot{r} = P_1 - \mu \sin\theta$ and $y = r\dot{\theta} = \frac{p_2}{r} - r - \mu \cos\theta$. Then

$$\begin{aligned} \omega' &= dr \wedge d\left(x + \mu \sin\theta\right) + d\theta \wedge d\left(ry + r^2 + r\mu \cos\theta\right) \\ &= dr \wedge dx + r d\theta \wedge dy + (-y - 2r) dr \wedge d\theta. \end{aligned}$$

And the Hamiltonian is

$$H' = \frac{x^2}{2} + \frac{y^2}{2} - \frac{r^2}{2} - \frac{1}{r} + \mu\left(-\mu/2 + \frac{1}{r} - r\cos\theta - \frac{1}{\sqrt{r^2+1+2r\cos\theta}}\right)$$

We introduce now $v = r^{1/2}x$ and $u = r^{1/2}y$, obtaining

$$\tilde{\omega} = r^{-1/2}dr \wedge dv + r^{1/2}d\theta \wedge du + \left(-r^{-1/2}\frac{u}{2} - 2r\right) dr \wedge d\theta.$$

And the top wedge is

$$\tilde{\omega}^2 = 2dr \wedge dv \wedge d\theta \wedge du.$$

If we introduce the time change to the geometric structure then

$$\omega_0 = r^{-2}dr \wedge dv + r^{-1}d\theta \wedge du + \left(-r^{-2}\frac{u}{2} - 2r^{-1/2}\right) dr \wedge d\theta.$$

And the top wedge is

$$\omega_0^2 = 2r^{-3}dr \wedge dv \wedge d\theta \wedge du.$$

This is a b^3-symplectic form. The final Hamiltonian is

$$\tilde{H} = \frac{v^2}{2r} + \frac{u^2}{2r} - \frac{r^2}{2} - \frac{1}{r} + \mu\left(-\mu/2 + \frac{1}{r} - r\cos\theta - \frac{1}{\sqrt{r^2+1+2r\cos\theta}}\right).$$

A result on ejection-collision orbits. Using the new Poisson structure, we obtain exactly the same equations as in [17]. Imposing $\iota_{X_H}\omega = dH$, the equations for the flow of X_H are:

$$\begin{aligned}
\dot{r} &= rv \\
\dot{v} &= \tfrac{v^2}{2} + u^2 - 1 + 2ur^{3/2} + r^3 + \mu \\
&\quad + \mu r^2(\cos\theta - (r+\cos\theta)(r^2+1+2r\cos\theta)^{-3/2}) \\
\dot{\theta} &= u \\
\dot{u} &= -\tfrac{uv}{2} - 2r^{3/2}v + \mu r^2 \sin\theta((r^2+1+2r\cos\theta)^{-3/2} - 1).
\end{aligned}$$

Observe that $M = -r^{-1/2}u - r^2$, the sidereal angular momentum, is conserved for $\mu = 0$. We can use the Poisson structure to compute its derivative for any μ:

$$\begin{aligned}
\dot{M} &= \frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial u}\dot{u} \\
&= -(\tfrac{u}{2\sqrt{r}} + 2r)rv - \sqrt{r}(-\tfrac{uv}{2} - 2r^{3/2}v + \mu r^2\sin\theta((r^2+1+2r\cos\theta)^{-3/2} - 1) \\
&= -\mu r^{5/2}\sin\theta((r^2+1+2r\cos\theta)^{-3/2} - 1).
\end{aligned}$$

Compute now the solutions for $\mu = 0$:

$$\begin{aligned}
r_0(\xi) &= 2(2C\cosh^2(\xi/\sqrt{2})^{-1} \\
v_0(\xi) &= -\sqrt{2}\tanh(\xi/\sqrt{2}) \\
\theta_0(\xi) &= -2C^{-3/2}(\sinh(\xi/\sqrt{2})\cosh^{-2}(\xi/\sqrt{2}) + \arctan[\sinh(\xi/\sqrt{2})) + \bar{\theta}.
\end{aligned}$$

Take $s = \xi/\sqrt{2}$ for simplicity. If $\dot{M} = \mu A(\theta)$, then after some computations the function A in the orbits for $\mu = 0$ is

$$A(\theta) = K \sin\theta C^{-5/2} \cosh^{-5}(s)(\Delta^{-3/2} - 1),$$

where K is a constant and $\Delta = 1 + \frac{4\cos\theta}{C\cosh^2 s} + \frac{4}{C^2\cosh^4 s}$. For C sufficiently large we have that $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. Doing the expansion of $(\Delta^{-3/2} - 1)$ we obtain

$$A(\bar{\theta}) = K' C^{-7/2} \bar{\theta} \cosh^{-7}(s) + \mathcal{O}(C^{-9/2}).$$

Now, in a first order and for C sufficiently large the integral of $\frac{\partial A(\bar{\theta})}{\partial \bar{\theta}}$ is

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\partial A(\bar{\theta})}{\partial \bar{\theta}} ds &= K' C^{-7/2} \int_{-\infty}^{+\infty} \cosh^{-7}(s) ds \\ &= K' C^{-7/2} \frac{5\pi}{16} \neq 0 \end{aligned}$$

In particular, we deduce that the sidereal angular momentum $M(\bar{\theta})$ has a simple zero at $\bar{\theta} = 0$. We then have orbits where the sidereal angular momentum is not conserved, and hence the orbit exits the collision manifold $\{r = 0\}$. This implies the theorem proved in [17] using different methods, in a shorter way and with less computations.

Theorem 46. *The restricted problem has transversal ejection-collision orbits for values of the mass parameter μ small enough and of the Jacobian constant C large enough.*

10 Conclusion

In the first sections of this work, we made a review of basic notions in symplectic and Poisson geometry. As we explained, development of Poisson geometry is done by generalizing main concepts of the symplectic world. However Poisson manifolds are much more general and hence more rich in structure and examples. A particular case of Poisson structure that has been specially interesting and hence detailed in the thesis was the b-Poisson one. Its dual counterpart are folded symplectic manifolds: the structures we have been mostly interested here.

Keeping in mind that we want to find similar results to the ones in symplectic geometry but with more general structures, we analysed the concept of integrable system. It is clearly defined and described semi-locally via an action-angle theorem theorem for the Poisson and b-symplectic case. This was first done in symplectic geometry in the socalled Arnold-Liouville theorem. Applying different methods from the classical ones we provided a new proof of the topological aspect of this action-angle coordinate theorem for both the symplectic and Poisson case. A few open problems resulted from this work, and we are still working on them. The generalized Tischler theorem that we proved may also be true for b-forms. This could lead to apply this new methods also in the b-symplectic case, as well as the folded symplectic case by duality.

In the folded symplectic world there was not a proper definition of integrable system nor a proof of existence of action-angle coordinates, so this was our next step. We analysed in detail how an integrable system could be defined such that it was well defined and had similar properties to the symplectic or Poisson case. Once properly defined, a singular Arnold-Liouville theorem was proved for these folded integrable systems. To obtain this result we first proved a folded version of Darboux-Carathéodory theorem, using a Moser's path method for folded structures. In this part we also had some more ideas and questions that could not be studied due to the size of this work, that we will now comment. Some of them will also be developped during the next months. For the whole theory of b-symplectic manifolds, the existence of a particular cotangent bundle was very fruitful and made possible a whole list of interesting constructions. This bundle called b-cotangent bundle makes it possible to work with differential form in a b-manifold. Since folded symplectic forms are somehow dual spaces to b-symplectic manifolds, we can also define a new cotangent bundle where forms vanish at the hypersurface. This idea of a new bundle in the folded case has been approached in [14]. It is only a first result on the existence of such a bundle but a whole theory has to be developed, inspired by the construction in the b-manifold case. We will work on this in the next months after the defense of this thesis.

Another open and much more difficult problem appeared when defining integrable systems on folded symplectic manifolds. It is common in the dynamical system's geometrical theory to forget about the Hamiltonian function determining the dynamics, and hence focusing on the geometric structures of the phase space. This was also done when doing some assumptions for the folded integrable systems. It remains difficult and an open question to keep track of the Hamiltonian and the dynamics on the level sets for the singular symplectic structures.

Another field where a lot of work still has to be done is the application to Celestial Mechanics. The singular geometric structures that we studied appear in a lot of different work on the n-body problem. What we proved in this thesis concerning Celestial Mechanics is just a small example of why it is interesting to keep track of the geometric structure when studying collisions. There are some of this versions where the geometric analysis has not been done yet, but also the increasing knowledge of this area of geometry needs to be more applied to this context.

11 Bibliography

References

[1] V. I. Arnold. *Remarks on Poisson structures on a plane and on other powers of volume elements*, (Russian) Trudy Sem. Petrovsk. No. 12 (1987), 3746, 242; translation in J. Soviet Math. 47 (1989), no. 3, 25092516.

[2] V.I. Arnold. *Critical point of smooth functions*, Vancouver Intern. Congr. of Math., 1974, vol.1, 1939.

[3] V.I. Arnold (1989). Mathematical Methods of Classical Mechanics. Springer (1989) ISBN 9780387968902.

[4] R. Braddell, A. Delshams, E. Miranda, C. Oms, A. Planas. *An invitation to singular symplectic geometry*, preprint 2017 arxiv:1705.03846.

[5] A. Cannas da Silva. *Lectures on Symplectic Geometry*, Springer-Verlag number 1764 of Lecture Notes in Mathematics series, revised January 2006.

[6] A. Cannas da Silva, V. Guillemin, and C. Woodward. *On the unfolding of folded symplectic structures.* Math. Res. Lett., 7(1):3553, 2000.

[7] R.Cardona, *Symplectic toric manifolds, Delzant Theorem and integrable systems.* Bachelor thesis at Universitat Politècnica de Catalunya supervised by E. Miranda and C. Oms, July 2017.

[8] R. Cardona, E. Miranda. *Integrable systems and closed one forms.* J. Geom. Phys. 131 (2018), 204-209.

[9] A. Delshams, A. Kiesenhofer, E. Miranda. *Examples of integrable systems and non-integrable systems on singular symplectic manifolds*, Journal of Geometry and Physics, 2016 DOI: 10.1016/j.geomphys.2016.06.011.

[10] V. Guillemin, E. Miranda, J. Weitsman. *Desingularizing b^m-symplectic structures*, International Mathematics Research Notices, rnx126, https://doi.org/10.1093/imrn/rnx126.

[11] C. Ehresmann. *Les connexions infinitésimales dans un espace fibré différentiable*, Colloque de Topologie, Bruxelles 29-55, 1950.

[12] R.L. Fernandes, I.Marcut. *Lectures in Poisson Geometry*, Springer (2015).

[13] V. Guillemin, E. Miranda, and A. Pires. *Symplectic and Poisson geometry on b-manifolds*, Adv. Math. 264 (2014), 864-896.

[14] D. Hockensmith. *A classification of toric, folded-symplectic manifolds*, PhD Thesis supervised by E. Lerman, August 2015.

[15] A. Kiesenhofer. *Integrable systems on b-symplectic manifolds*, PhD Thesis supervised by E. Miranda, October 2016.

[16] A. Kiesenhofer, E. Miranda, G. Scott. *Action-angle variables and a KAM theorem for b-Poisson manifolds* , Journal des Mathmatiques Pures et Appliques, J. Math. Pures Appl. (9) 105 (2016), no. 1, 66-85.

[17] E.A. Lacomba, J. Llibre.*Transversal ejection-collision orbits for the restricted problem and the Hill's problem with applications*, Journal of Differential Equations, Volume 74, Issue 1, p. 69-85 (1998).

[18] C. Laurent-Gengoux, E. Miranda, P. Vanhaecke. *Action-angle Coordinates for Integrable Systems on Poisson Manifolds*, International Mathematics Research Notices, Volume 2011, Issue 8, 1 January 2011, pages 18391869, https://doi.org/10.1093/imrn/rnq130.

[19] J. Martinet, *Sur les singularites des formes différentielles.* Ann. Inst. Fourier (Grenoble) 20 1970 fasc. 1, 95178.

[20] R. McGehee. *Double collisions for a classical particle system with nongravitational interactions*, Comment. Math. Helvetici 56 (1981) 524-557.

[21] R. Melrose. *Atiyah-Patodi-Singer Index Theorem* (book) Research Notices in Mathematics, A.K. Peters, Wellesley, 1993.

[22] A.R. Pires. *Origami manifolds* , thesis (Ph. D.) Massachusetts Institute of Technology supervised by V. Guillemin, 2010.

[23] A.Planas, E.Miranda. *Equivariant classification of b^m-symplectic surfaces*, https://arxiv.org/abs/1607.01748, to appear at Regular and Chaotic Dynamics, Volume 23, Issue 4 of 2018.

[24] G. Scott, *The Geometry of b^k Manifolds*, J. Symplectic Geom. 14 (2016), no. 1, 7195.

[25] P. Stefan. *Accessible sets, orbits, and foliations with singularities.* Proc. London Math. Soc., s3-29(4):699713, 1974.

[26] H. Sussmann. *Orbits of families of vector fields and integrability of distributions.* Trans. Amer. Math. Soc., 180:171188, 1973.

[27] D. Tischler. *On fibering certaing foliated manifolds over S^1* , Topology 9 153-154, 1970.

[28] A. Weinstein and Conference Board of the Mathematical Sciences. *Lectures on symplectic manifolds.* Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society (1977), Providence. R.I

[29] A. Zamorzaev. *Unofficial notes for Introduction to Symplectic Topology*, Stanford University, Autumn 2012.